Global controllability of Boussinesq flows by using only a temperature control

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Abstract

We show that buoyancy driven flows can be steered in an arbitrary time towards any state by applying as control only an external temperature profile in a subset of small measure. More specifically, we prove that the 2D incompressible Boussinesq system on the torus is globally approximately controllable via physically localized heating or cooling. In addition, our controls have an explicitly prescribed structure; even without such structural requirements, large data controllability results for Boussinesq flows driven merely by a physically localized temperature profile were so far unknown. The presented method exploits various connections between the model's underlying transport-, coupling-, and scaling mechanisms.

Keywords
Boussinesq system, incompressible fluids, controllability, global approximate controllability, decomposable controls, transported Fourier modes

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35Q30, 35Q49, 76B75, 80A19, 93B05, 93C10

1 Introduction

We demonstrate the global approximate controllability of incompressible buoyancy-driven flows regulated only by a physically localized temperature control; "global" means in this context that the initial and target profiles might be far apart from each other in the state space. The considered incompressible Boussinesq system is relevant to the study of, *e.g.*, geophysical phenomena and Rayleigh-Bénard convection, and it also serves applications involving heating and ventilation (*cf.* [1, 14, 23]). In particular, it is desirable to uncover coupling mechanisms that facilitate the controllability of

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nonlinear fluids merely via regionally applied heating/cooling, without imposing smallness constraints on the data. This can matter from practical perspectives and provides deeper theoretical insights regarding the mathematical model itself. But despite the variety of motivations, all available controllability results for the Boussinesq equations steered only by a temperature control have been limited to small perturbations of linear dynamics in 2D; see Section 1.5 for bibliographical remarks. In this article, contrasting the existing literature, a truly global controllability problem for the nonlinear Boussinesq system is tackled; the present approach even allows to fix the control's structure in terms of a small finite number of universal physically localized profiles. Our proof features geometric arguments, a multi-stage scaling procedure, and the notion of transported Fourier modes from [20].

1.1 The main controllability problem

Let $T > 0$ and assume that the gravitational field is given by $\boldsymbol{e}_{\mathrm{grav}} := [0,1]^\top$. The state of a viscous incompressible fluid in $\mathbb{T}^2 := \mathbb{R}^2/2\pi\mathbb{Z}^2$ is then described by means of its 2π-periodic velocity, temperature, and exerted pressure; respectively,

$$\boldsymbol{u}\colon \mathbb{T}^2 \times [0,T] \longrightarrow \mathbb{R}^2, \quad \theta\colon \mathbb{T}^2 \times [0,T] \longrightarrow \mathbb{R}, \quad p\colon \mathbb{T}^2 \times [0,T] \longrightarrow \mathbb{R},$$

which are governed by the Boussinesq system

$$\begin{gathered} \partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} + \boldsymbol{\nabla} p = \theta\boldsymbol{e}_{\mathrm{grav}} + \boldsymbol{\Phi}_{\mathrm{ext}}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \quad \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0,\\ \partial_t\theta - \tau\Delta\theta + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\theta = \mathbb{I}_{\omega}\eta + \psi_{\mathrm{ext}}, \quad \theta(\cdot,0) = \theta_0, \end{gathered} \tag{1.1}$$

where $\nu > 0$ is the viscosity and $\tau > 0$ denotes the thermal diffusivity. Moreover, the functions $\boldsymbol{\Phi}_{\mathrm{ext}}$ and ψ_{ext} represent given external forces, which are assumed to be average-free for simplicity. A distinguished role is played by the unknown profile η in (1.1); it will act as the control and shall be localized in an arbitrarily thin horizontal strip (*cf.* Figure 1)

$$\omega := \mathbb{T} \times (a,b), \quad 0 < a < b \leq 2\pi.$$

Given any initial states $(\boldsymbol{u}_0, \theta_0)$, target states $(\boldsymbol{u}_T, \theta_T)$, control time $T > 0$, and approximation error $\varepsilon > 0$, we show (*cf.* Theorem 1.2) that there exists a temperature control η supported in ω such that the corresponding solution $(\boldsymbol{u}, \theta)$ to (1.1) approaches the target at time $t = T$ with respect to a Sobolev norm $\|\cdot\|$; that is,

$$\|\boldsymbol{u}(\cdot,T) - \boldsymbol{u}_T\| + \|\theta(\cdot,T) - \theta_T\| < \varepsilon. \tag{1.2}$$

1.2 More degenerate controllability problems

Our goal described above will be achieved as by-product of studying a more degenerate situation. Hereto, let us consider the Boussinesq problem with an additional control acting parallel to gravitation on the velocity; namely,

$$\begin{gathered} \partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} + \boldsymbol{\nabla} p = (\theta + \mathbb{I}_{\omega}\overline{\eta})\boldsymbol{e}_{\mathrm{grav}} + \boldsymbol{\Phi}_{\mathrm{ext}}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \quad \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0,\\ \partial_t\theta - \tau\Delta\theta + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\theta = \mathbb{I}_{\omega}\eta + \psi_{\mathrm{ext}}, \quad \theta(\cdot,0) = \theta_0. \end{gathered} \tag{1.3}$$

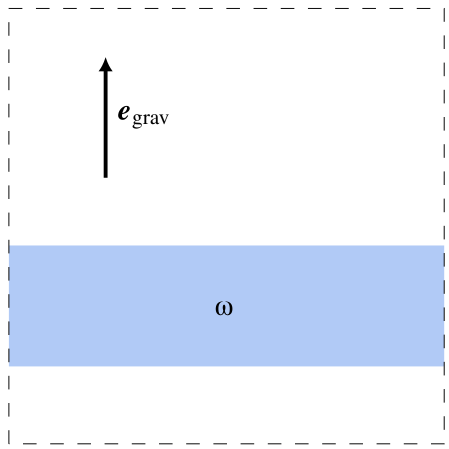

Figure 1: The control region $\omega \subset \mathbb{T}^2$ is any open horizontal strip and $\boldsymbol{e}_{\text{grav}}$ points vertically.

We shall obtain controls η and $\overline{\eta}$ that are supported in ω, composed of a few universal profiles, and ensure for the respective solution to (1.3) the property

$$\|\boldsymbol{u}(\cdot, T) - \boldsymbol{u}_T\| + \|\theta(\cdot, T) - \theta_T\| < \varepsilon. \tag{1.4}$$

In other words, we are going to describe a small number of universally fixed profiles (actuators) $\zeta_1, \dots, \zeta_6 \in \mathrm{L}^2((0,1); \mathrm{C}^\infty(\mathbb{T}^2; \mathbb{R}))$ that vanish outside ω and are independent of all imposed data – except the control region ω – such that

$$\eta(\boldsymbol{x}, t) = \sum_{l=1}^{6} \gamma_l(t)\zeta_l(\boldsymbol{x}, \gamma(t)), \quad \overline{\eta}(x_1, x_2, t) = \overline{\gamma}(t)\zeta_1(x_2), \tag{1.5}$$

where the unknown parameters $\gamma, \overline{\gamma}, \gamma_1, \dots, \gamma_6 \in \mathrm{L}^2((0,T); \mathbb{R})$ resemble the actual controls; we call this type of control "finitely decomposable".

Notably – when one varies the initial and target states, the viscosity, the thermal diffusivity, the external forces, or the error – the profiles $\zeta_1, \dots, \zeta_6$ in (1.5) remain unchanged. Thus, solving the controllability problem (1.3)-(1.4) means determining $\overline{\gamma}, \gamma_1, \dots, \gamma_6$, and γ such that the solution to (1.3) obeys (1.4). In fact, $\overline{\gamma}$ will be smooth with $\operatorname{supp}(\overline{\gamma}) \subset (0, T)$, and it shall be possible to take ζ_1 and ζ_2 as single variable functions satisfying $\zeta_2 = \zeta_1'$. Further, as detailed in Section 3.3, one may, by means of the transformation $\theta(x_1, x_2, t) \mapsto \theta(x_1, x_2, t) + \overline{\gamma}(t)\zeta_1(x_2)$, interchange the controls in (1.5) with $\overline{\eta} = 0$ and a finitely decomposable temperature control plus explicit feedback term; that is,

$$\begin{aligned}
\eta(x_1, x_2, t) &= \left(\overline{\gamma}'(t) + \gamma_1(t)\right)\zeta_1(x_2) + \left(\overline{\gamma}(t)\boldsymbol{u}(x_1, x_2, t) \cdot \boldsymbol{e}_{\text{grav}} + \gamma_2(t)\right)\zeta_2(x_2) \\
&\quad - \tau\overline{\gamma}(t)\zeta_1''(x_2) + \sum_{l=3}^{6} \gamma_l(t)\zeta_l(x_1, x_2, \gamma(t)), \\
\overline{\eta}(x_1, x_2, t) &= 0,
\end{aligned} \tag{1.6}$$

noting that η could in (1.3) be replaced by a smooth approximation of η while maintaining (1.4). In (1.6), the map $\boldsymbol{u} \mapsto \overline{\gamma}\zeta_2\boldsymbol{u} \cdot \boldsymbol{e}_{\text{grav}}$ can be viewed as an explicitly given linear feedback law for the universally given actuator ζ_2, which is independent of the imposed data. The main controllability problem described in Section 1.1, where only a physically localized temperature control is employed, is solved by finding a control η of the form stated in (1.6) for the Boussinesq problem (1.3).

Remark 1.1. As a side-note (*cf.* Remark 3.8), when allowing a non-localized spatially constant profile in the temperature control, we can also ensure (1.2) by using merely a finitely decomposable temperature control; *i.e.*, replacing $\mathbb{I}_\omega \eta(\boldsymbol{x}, t)$ in (1.1) by

$$\overline{\gamma} + \mathbb{I}_\omega(\gamma_1(t)\zeta_1(\boldsymbol{x}, \gamma(t)) + \cdots + \gamma_6(t)\zeta_6(\boldsymbol{x}, \gamma(t))).$$

While the physical localization of one actuator (which is here a constant function) is lost in that case, achieving such a representation still extends considerably the existing literature on degenerate controls for incompressible fluids.

Another interest of our work is to relax a topological constraint previously imposed on the control region in [20], where the two-dimensional Navier–Stokes system with physically localized finitely decomposable controls is considered. There, in order to act on the velocity-average, the control zone is required to contain two cuts rendering the torus simply-connected (*e.g.*, the union of two strips with linearly independent direction vectors). In the present article, by exploiting the buoyant force, coupling the velocity and temperature in the momentum equation, we are able to take ω merely as a horizontal strip. This observation indicates that heat effects in the mathematical model might improve certain controllability properties.

1.3 Notations

Given any integer $m \geq 0$, several basic L^2-based Sobolev spaces of average-free functions and divergence-free vector fields are denoted by

$$\mathrm{H}_{\mathrm{avg}} := \left\{ f \in \mathrm{L}^2(\mathbb{T}^2;\mathbb{R}) \mid \int_{\mathbb{T}^2} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = 0 \right\}, \; \mathbf{H}_{\mathrm{div}} := \left\{ \boldsymbol{f} \in \mathrm{L}^2(\mathbb{T}^2;\mathbb{R}^2) \mid \boldsymbol{\nabla} \cdot \boldsymbol{f} = 0 \right\},$$
$$\mathbf{V}^m := \mathrm{H}^m(\mathbb{T}^2;\mathbb{R}^2) \cap \mathbf{H}_{\mathrm{div}}, \quad \mathbf{H}^m := \mathbf{V}^m \cap \mathrm{H}^2_{\mathrm{avg}}, \quad \mathrm{H}^m := \mathrm{H}^m(\mathbb{T}^2;\mathbb{R}) \cap \mathrm{H}_{\mathrm{avg}},$$

endowed with the norms

$$\|\cdot\|_m := \sqrt{\sum_{|\alpha| \leq m} \|\partial^\alpha \cdot\|_0^2}, \quad \|\cdot\|_0 := \text{ either } \|\cdot\|_{\mathrm{L}^2(\mathbb{T}^2;\mathbb{R})} \text{ or } \|\cdot\|_{\mathrm{L}^2(\mathbb{T}^2;\mathbb{R}^2)},$$

and the Lebesgue measure on the torus is assumed normalized: $\int_{\mathbb{T}^2} \mathrm{d}\boldsymbol{x} = 1$. The symbol $\mathcal{O}$ refers to the Landau-big-O notation. If not specified otherwise, constants of the form $C > 0$ are unessential and their values may vary during estimates.

1.4 Main results

Throughout, we shall employ a universally fixed collection of so-called transported Fourier modes $\zeta_1, \ldots, \zeta_6 \in \mathrm{L}^2((0,1); \mathrm{C}^\infty(\mathbb{T}^2;\mathbb{R}))$ as the building blocks for the controls described in (1.5); they shall be determined during the proof of Theorem 2.7 in a way only depending on ω (Figure 2), and their name is due to the involved composition of usual Fourier modes with certain flow maps. More specifically, (*cf.* (2.7) and (2.20))

$$\{\zeta_1, \dots, \zeta_6\} = \{\chi, \chi'\} \cup \left\{(\boldsymbol{x}, t) \mapsto \chi(\boldsymbol{x})\widetilde{\zeta}(\boldsymbol{\mathcal{U}}(\boldsymbol{\mathcal{Y}}(\boldsymbol{x}, t, 1), 1, \sigma(t))) \mid \widetilde{\zeta} \in \mathscr{M}\right\},$$

where

- $\mathscr{M} = \{\boldsymbol{x} \mapsto \sin(x_1), \boldsymbol{x} \mapsto \cos(x_1), \boldsymbol{x} \mapsto \sin(x_2), \boldsymbol{x} \mapsto \cos(x_2)\}$,
- $\boldsymbol{\mathcal{Y}}$ and $\boldsymbol{\mathcal{U}}$ are the flows of incompressible vector fields from Section 2.1,
- $\sigma\colon [0,1] \longrightarrow [0,1]$ is the function defined in (2.19),
- $\boldsymbol{x} \mapsto \chi(x_2)$ is a smooth cutoff supported in ω introduced in Section 2.1.

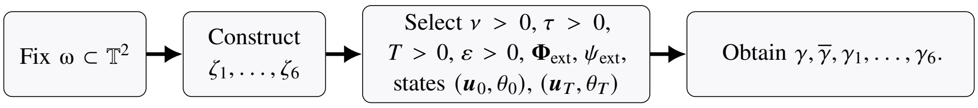

Figure 2: The order in which the building blocks for our control force are chosen.

The global approximate controllability of (1.1) by means of a physically localized temperature control is now stated as follows; its proof is concluded in Section 3.3.

Theorem 1.2. *Let the integer $r \geq 0$, viscosity $\nu > 0$, diffusivity $\tau > 0$, control time $T > 0$, initial and target states $\boldsymbol{u}_0, \boldsymbol{u}_T \in \mathbf{H}^r$, $\theta_0, \theta_T \in \mathrm{H}^r$, external forces $(\boldsymbol{\Phi}_{\mathrm{ext}}, \psi_{\mathrm{ext}}) \in \mathrm{L}^2((0,T); \mathbf{H}^{\max\{r,2\}} \times \mathrm{H}^{\max\{r,2\}})$, and the approximation error $\varepsilon > 0$ be fixed. There exists a control $\eta \in \mathrm{C}^\infty([0,T]; \mathrm{C}^\infty(\mathbb{T}^2;\mathbb{R}))$ with $\operatorname{supp}(\eta) \subset \omega$ such that the unique solution*

$$\boldsymbol{u} \in \mathrm{C}^0([0,T]; \mathbf{V}^r) \cap \mathrm{L}^2((0,T); \mathbf{V}^{r+1}),$$
$$\theta \in \mathrm{C}^0([0,T]; \mathrm{H}^r(\mathbb{T}^2;\mathbb{R})) \cap \mathrm{L}^2((0,T); \mathrm{H}^{r+1}(\mathbb{T}^2;\mathbb{R}))$$

to the Boussinesq problem (1.1) *satisfies*

$$\|\boldsymbol{u}(\cdot,T) - \boldsymbol{u}_T\|_r + \|\theta(\cdot,T) - \theta_T\|_r < \varepsilon. \tag{1.7}$$

The proof of Theorem 1.2 is a consequence of the following result on the global approximate controllability of the Boussinesq system via physically localized finitely decomposable controls; both theorems are proved in Section 3.3.

Theorem 1.3. *Under the same assumptions as in Theorem 1.2, there exist control parameters $\gamma, \overline{\gamma}, \gamma_1, \dots, \gamma_6 \in \mathrm{L}^2((0,T);\mathbb{R})$ such that the unique solution*

$$\boldsymbol{u} \in \mathrm{C}^0([0,T]; \mathbf{V}^r) \cap \mathrm{L}^2((0,T); \mathbf{V}^{r+1}),$$
$$\theta \in \mathrm{C}^0([0,T]; \mathrm{H}^r(\mathbb{T}^2;\mathbb{R})) \cap \mathrm{L}^2((0,T); \mathrm{H}^{r+1}(\mathbb{T}^2;\mathbb{R}))$$

to the problem (1.3) *with η and $\overline{\eta}$ of the form* (1.5) *(or* (1.6)*) satisfies* (1.7).

1.5 Related literature and outline

The recent decades have seen various studies concerned with controllability properties of fluids exhibiting Boussinesq heat effects. A natural question – which remains in most respects widely open – is whether these systems can be steered to a desired state by merely applying external cooling or heating in a possibly small subset of the domain. Let us subdivide previous research in that or related directions into three categories.

1) When the controls enter all the evolution equations (or boundary conditions) for the velocity and the temperature, there exists a rich body of literature. For instance, several authors have invoked linearization techniques and then studied the controllability of linear problems; *e.g.*, see [13, 15] and the references therein, where duality arguments, Carleman estimates, and local inversion theorems play crucial roles. But, due to the nonlinear effects, the aforementioned results require initial and target profiles that are sufficiently close in the state space. As a way to remove such smallness constraints, hence to achieve global controllability properties, it was shown that Coron's return method (*cf.* [8]) can be applied: *e.g.*, in [6, 12, 13] for both viscous and inviscid Boussinesq systems. In this context, one should also name a famous open problem posed by J.-L. Lions on the global approximate controllability of the Navier–Stokes equations, in bounded 2D and 3D domains, with the no-slip boundary condition (*cf.* [10, 17]).

2) When the controls only act in few components of the considered system, less is known regarding its controllability. For the three-dimensional Navier–Stokes equations with the no-slip boundary condition, the work [9] demonstrates the local exact null controllability with controls vanishing in two components; even in periodic domains, and for 2D configurations, the global exact null controllability through few components in fixed time $T > 0$ is an open problem (see also the bibliography of [9] for related results). For the Boussinesq system in N dimensions ($N \geq 2$), the local exact controllability to certain trajectories has been shown by using controls that act only in $N - 1$ directions, *e.g.*, in [5, 11, 18].

3) Another important class of controls are those resembling finite combinations of fixed actuators. For the Navier–Stokes equations, finite-dimensional controls can be constructed via the Agrachev-Sarychev method [3] and its refinements, for instance, as provided in [19, 21]. However, in these references, the controls are not physically localized (they act everywhere in the torus); whether their localization in space is possible constitutes an open problem due to Agrachev [2]. To achieve also physical localization, we replaced in [20] the notion of finite-dimensional controls by that of finitely decomposable ones, where some of the universally fixed actuators depend on time. Here, as a byproduct, we extend these results to the 2D Boussinesq system. By exploiting the temperature coupling, this leads now to additional improvements such as reduced constraints on the control region (in [20], the controls are not supported in a horizontal strip) and the possibility of using in the velocity equation a one-dimensional control with one vanishing component.

This article contributes a first global (large data) controllability result for the

Boussinesq system via physically localized controls acting only in the temperature. Even more, the controls can be chosen finitely decomposable (of the type (1.5)) if one admits a one-dimensional control in the second component of the velocity problem. But, up to an explicit feedback supported in ω, we can also steer the system merely with a physically localized and finitely decomposable temperature control (*cf.* (1.6)). Moreover, we allow prescribed external forces in the right-hand sides of the Boussinesq problem. The main ingredients of our approach are as follows.

a) A multi-stage scaling procedure that combines two mechanisms: i) controlling the fluid's vorticity, but ignoring the temperature; ii) steering the temperature without influencing the vorticity much. To this end, we develop ideas from [4, 20], and also involve a version of Coron's return method and hydrodynamic scaling from [7].

b) The physical localization of $\zeta_1, \ldots, \zeta_6$ is achieved via careful rearrangements of integrals that represent solutions to transport equations; in that way, we further develop several of our ideas from [20].

Outline of the paper. As described in Section 3, the proofs of Theorems 1.2 and 1.3 reduce to controlling the vorticity, the temperature, and the average velocity. These sub-goals are achieved by means of the following main steps (*cf.* Figure 3). *A)* The temperature can be controlled without significantly changing the vorticity; see Theorem 3.3. Hereto, preliminary constructions are presented in Section 2.1, finitely decomposable controls (possibly supported everywhere) are obtained in Section 2.2 for linear equations, a localization procedure is carried out in Section 2.3, and a hydrodynamic scaling is discussed in Section 3.1. *B)* As stated in Theorem 3.4, the vorticity can be controlled through well-prepared initial conditions. *C)* The previous arguments are put together in Theorem 3.7 for the vorticity-temperature formulation, and in Section 3.3 for the velocity-temperature problem.

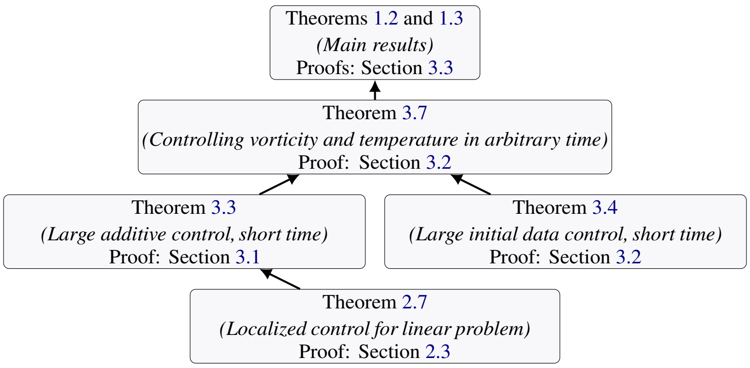

Figure 3: Subdivision of the proofs for Theorems 1.2 and 1.3.

2 Controls for linear transport type problems

This section concerns linear transport problems driven by finitely decomposable controls. We first obtain controls supported in $\mathbb{T}^2$, then further localize them in ω.

2.1 Preliminary constructions

This section aims to collect several technical definitions that are required subsequently.

2.1.1 Partition of the torus

Select $0 < H_1 < H_2 < 2\pi$ in a way that $\mathbb{T} \times [H_1, H_2] \subset \omega$. Then, a number $K \in \mathbb{N}$ is chosen such that

$$l_K \coloneqq \frac{8\pi}{3K} < \frac{H_2 - H_1}{3}.$$

As illustrated in Figure 4, the torus $\mathbb{T}^2$ may thus be covered by a family of overlapping strips $(O_i)_{i\in\{1,\dots,K\}}$ having fixed overlap length $l_K/4$ and being translated copies of the reference strip

$$O \coloneqq \mathbb{T} \times (H_1 + l_K, H_1 + 2l_K) \subset \omega.$$

For definiteness, let us take

$$O_i \coloneqq \mathbb{T} \times \left(\frac{3(i-1)l_K}{4}, \frac{3(i-1)l_K}{4} + l_K\right), \quad i \in \{1, \dots, K\}.$$

On this basis, a reference cutoff function $\chi \in \mathrm{C}^\infty(\mathbb{T}^2; [0,1])$ with $\operatorname{supp}(\chi) \subset O$ is specified via

$$\chi(\boldsymbol{x}) = \chi(x_2) \coloneqq \mu(x_2 - H_1 - l_K), \quad \boldsymbol{x} = [x_1, x_2]^\top \in \mathbb{T}^2, \tag{2.1}$$

where $\mu \in \mathrm{C}^\infty(\mathbb{T}; [0,1])$ satisfies

$$\begin{gathered} \operatorname{supp}(\mu) \subset (0, l_K), \quad \forall x \in (0, l_K/4)\colon \mu(x) + \mu(x + 3l_K/4) = 1, \\ \mu(s) = 1 \iff s \in [l_K/4, 3l_K/4]. \end{gathered} \tag{2.2}$$

2.1.2 Convection strategy

For the purpose of localizing in Section 2.3 certain controls for linear transport problems in the control zone ω, a spatially constant vector field is now constructed so that all its associated integral curves pass through ω in a specific way. This profile will enable us later to utilize Coron's return method (*cf.* [8]); we already introduced similar constructions in [20]. To begin with, the reference time interval $[0, 1]$ is equidistantly partitioned into subintervals of length $T^\Delta > 0$ by means of

$$0 < t_c^0 < t_a^1 < t_b^1 < t_c^1 < t_a^2 < t_b^2 < t_c^2 < \dots < t_a^K < t_b^K < t_c^K < 1, \tag{2.3}$$

where $t_c^0 = t_a^i - t_c^{i-1} = t_c^i - t_b^i = t_b^i - t_a^i = 1 - t_c^K = T^\Delta$ for all $i \in \{1, \dots, K\}$.

Theorem 2.1. *There exists a function $t \mapsto \overline{\boldsymbol{y}}(t) = [0, \overline{y}_2(t)] \in \mathrm{C}_0^\infty((0,1); \mathbb{R}^2)$ that satisfies together with its flow $\boldsymbol{\mathcal{Y}}$, which is obtained by solving*

$$\frac{\mathrm{d}\boldsymbol{\mathcal{Y}}}{\mathrm{d}t}(\boldsymbol{x}, s, t) = \overline{\boldsymbol{y}}(t), \quad \boldsymbol{\mathcal{Y}}(\boldsymbol{x}, s, s) = \boldsymbol{x}, \tag{2.4}$$

the three properties listed below.

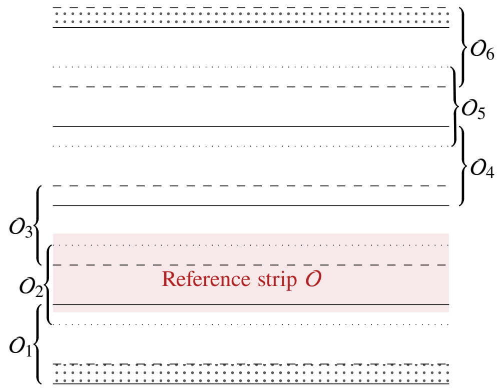

Figure 4: An exemplary open covering of $\mathbb{T}^2$ by $K = 6$ overlapping strips $O_1, \ldots, O_6$, the boundaries of which are in an alternating way depicted as solid, dashed, and dotted lines. The overlapping region due to vertical periodicity is highlighted by a dotted pattern. The reference strip O contained inside the control region is displayed as a (red) filled rectangle.

- *Supported in* $[t_c^0, t_c^K]$*:* $\forall t \in [0, t_c^0] \cup [t_c^K, 1] : \overline{\boldsymbol{y}}(t) = \mathbf{0}$.
- *Closed integral curves:* $\forall \boldsymbol{x} \in \mathbb{T}^2 : \boldsymbol{\mathcal{Y}}(\boldsymbol{x}, 0, 1) = \boldsymbol{x}$.
- *Stationary visits of* O*:* $\forall i \in \{1, \ldots, K\}$, $\forall t \in [t_a^i, t_b^i] : \ \boldsymbol{\mathcal{Y}}(O_i, 0, t) = O$.

Proof. The argument goes along the lines of [20, Theorem 3.3]. First, a collection of functions $(\beta_i)_{i\in\{1,\ldots,K\}} \subset \mathrm{C}_0^\infty((0, T^\Delta); \mathbb{R})$ is chosen in a way that

$$O_i + \boldsymbol{e}_{\text{grav}} \int_0^{T^\Delta} \beta_i(s) \, \mathrm{d}s = O.$$

Subsequently, for any $i \in \{1, \ldots, K\}$, a profile $\boldsymbol{h}_i \in \mathrm{C}_0^\infty((0, 3T^\Delta); \mathbb{R}^2)$ is piece-wise defined by means of

$$\boldsymbol{h}_i(t) = \begin{cases} \beta_i(t)\boldsymbol{e}_{\text{grav}} & \text{if } t \in [0, T^\Delta], \\ \mathbf{0} & \text{if } t \in (T^\Delta, 2T^\Delta), \\ -\beta_i(t - 2T^\Delta)\boldsymbol{e}_{\text{grav}} & \text{if } t \in [2T^\Delta, 3T^\Delta]. \end{cases}$$

Finally, after integrating (2.4), it follows that the profile

$$\overline{\boldsymbol{y}}(t) := \begin{cases} \mathbf{0} & \text{if } t \in [0, t_c^0] \cup [t_c^K, 1], \\ \boldsymbol{h}_i(t - (3i - 2)T^\Delta) & \text{if } t \in (t_c^{i-1}, t_c^i) \text{ for } i \in \{1, \ldots, K\} \end{cases}$$

satisfies the desired properties. □

2.1.3 A generating vector field

Let us setup some terminology that will be required later in Section 2.2. A key ingredient is the following observability notion, which has been introduced in [16] for the study of mixing properties of randomly forced PDEs.

Definition 2.2. Given any $T > 0$ and $n \in \mathbb{N}$, a family $(\phi_j)_{j\in\{1,\dots,n\}} \subset \mathrm{L}^2((0,T);\mathbb{R})$ is said to be observable if

$$\begin{gathered}\forall\, \mathrm{I} \in \{\text{subintervals of } (0,T)\},\ \forall\, (a_j)_{j\in\{1,\dots,n\}} \subset \mathrm{C}^1(\mathrm{I};\mathbb{R}),\ \forall\, b \in \mathrm{C}^0(\mathrm{I};\mathbb{R})\colon \\ b + \sum_{j=1}^{n} a_j\phi_j = 0 \text{ in } \mathrm{L}^2(\mathrm{I};\mathbb{R}) \quad \Longrightarrow \quad \forall t \in \mathrm{I}\colon b(t) = a_1(t) = \dots = a_n(t) = 0.\end{gathered}$$

Remark 2.3. Observable families in the sense of Definition 2.2 can be constructed in an explicit way; *e.g.*, see [19, Section 3.3] or [20] for more details.

Let $(\phi_l)_{l\in\{1,\dots,4\}} \subset \mathrm{L}^2((0,1);\mathbb{R})$ be observable, and take $\phi \in \mathrm{C}^1([0,1];\mathbb{R})$ such that $\phi(t) = 0$ if and only if $t = 1$. Furthermore, define the family

$$(\psi_l)_{l\in\{1,\dots,4\}} \subset \mathrm{W}^{1,2}((0,1);\mathbb{R}), \quad \forall l \in \{1,\dots,4\}\colon\ \psi_l(t) := \phi(t)\int_0^t \phi_l(s)\,\mathrm{d}s.$$

Then, a "generating" divergence-free vector field $\overline{\boldsymbol{u}} \in \mathrm{W}^{1,2}((0,1);\mathrm{C}^\infty(\mathbb{T}^2;\mathbb{R}^2))$ is given via (*cf.* [20, Section 3.4])

$$\overline{\boldsymbol{u}}(\boldsymbol{x},t) := \begin{bmatrix}\psi_1(t)\sin(x_2) + \psi_2(t)\cos(x_2)\\ \psi_3(t)\sin(x_1) + \psi_4(t)\cos(x_1)\end{bmatrix},$$

and its flow $\boldsymbol{\mathcal{U}}$ solves

$$\frac{\mathrm{d}\boldsymbol{\mathcal{U}}}{\mathrm{d}t}(\boldsymbol{x},s,t) = \overline{\boldsymbol{u}}(\boldsymbol{\mathcal{U}}(\boldsymbol{x},s,t),t), \quad \boldsymbol{\mathcal{U}}(\boldsymbol{x},s,s) = \boldsymbol{x}.$$

Here, the term "generating" expresses that such vector fields are able to induce all desired directions via finite-dimensional controls, as demonstrated in, *e.g.*, [19,20].

2.2 Degenerate non-localized controls

Let $\overline{\boldsymbol{y}}$ with flow $\boldsymbol{\mathcal{Y}}$ and $\overline{\boldsymbol{u}}$ with flow $\boldsymbol{\mathcal{U}}$ be as introduced in Sections 2.1.2 and 2.1.3, respectively. To begin with, we state the following result whose proof is straightforward.

Lemma 2.4. *Given $m \in \mathbb{N}$ and $\boldsymbol{b} \in \mathrm{C}^0([0,1];\mathrm{C}^m(\mathbb{T}^2;\mathbb{R}^2))$, the linear operator which associates to any prescribed force $g \in \mathrm{L}^2((0,1);\mathrm{H}^m(\mathbb{T}^2;\mathbb{R}))$ the unique solution $v \in \mathrm{C}^0([0,1];\mathrm{H}^m(\mathbb{T}^2;\mathbb{R}))$ to the transport equation*

$$\partial_t v + (\boldsymbol{b}\cdot\boldsymbol{\nabla})v = g, \quad v(\cdot,0) = 0$$

maps continuously from $\mathrm{L}^2((0,1);\mathrm{H}^m(\mathbb{T}^2;\mathbb{R}))$ to $\mathrm{C}^0([0,1];\mathrm{H}^m(\mathbb{T}^2;\mathbb{R}))$.

Next, a recent result from [20] is recalled concerning finite-dimensional controls for linear transport equations with a generating drift as defined above in Section 2.1.3. Hereby, as already anticipated in Section 1.4, we denote the four-dimensional function space

$$\mathcal{H}_0 = \operatorname{span}_{\mathbb{R}} \mathscr{M}, \quad \mathscr{M} = \{\sin(x_1),\, \cos(x_1),\, \sin(x_2),\, \cos(x_2)\}. \tag{2.5}$$

The following theorem can be verified by adopting [19, Section 2.3] (written there for 3D) to the 2D case; see [20] for specific details. Hereby, if desired, the control can be selected in continuous dependence on (v_1, θ_1) following a compactness argument as explained in [19, Proof of Theorem 2.3] or [16, Proposition 2.6].

Theorem 2.5. *Given any $m \in \mathbb{N}$, $z_1 \in \mathrm{H}^m$, and $\varepsilon > 0$, there exists $g \in \mathrm{L}^2((0,1);\mathcal{H}_0)$ such that the unique solution $z \in \mathrm{C}^0([0,1];\mathrm{H}^m) \cap \mathrm{W}^{1,2}((0,1);\mathrm{H}^{m-1})$ to the linear transport problem*

$$\partial_t z + (\overline{\boldsymbol{u}} \cdot \boldsymbol{\nabla}) z = g, \quad z(\cdot, 0) = 0 \tag{2.6}$$

obeys

$$\|z(\cdot,1) - z_1\|_m < \varepsilon.$$

In addition, given a bounded subset $\mathrm{B} \subset \mathrm{H}^m$, there exists a continuous linear operator C_ε which assigns to each $z_1 \in \mathrm{B}$ a control $g \in \mathrm{L}^2((0,T);\mathcal{H}_0)$ such that the corresponding solution z to (2.6) satisfies $\|z(\cdot,1) - z_1\|_{m-1} < \varepsilon$.

Now, we consider a linearized Boussinesq system driven by degenerate controls that are potentially supported everywhere in $\mathbb{T}^2$. To prepare the localization procedure carried out later in Section 2.3, convection will now be realized along $\overline{\boldsymbol{y}}$, and a finite family of transported Fourier modes $\{\widetilde{\zeta}_1, \ldots, \widetilde{\zeta}_4\} \subset \mathrm{L}^2((0,1);\mathrm{C}^\infty(\mathbb{T}^2;\mathbb{R}))$ shall be involved instead of $\mathcal{H}_0$; namely, we enumerate

$$\{\widetilde{\zeta}_1, \ldots, \widetilde{\zeta}_4\} = \left\{(\boldsymbol{x}, t) \mapsto \widehat{\zeta}(\boldsymbol{\mathcal{U}}(\boldsymbol{\mathcal{Y}}(\boldsymbol{x}, t, 1), 1, t)) \,\middle|\, \widehat{\zeta} \in \mathscr{M}\right\}. \tag{2.7}$$

The definition in (2.7) is motivated by the proof of the following theorem.

Theorem 2.6. *For any $m \in \mathbb{N}$, $\theta_1 \in \mathrm{H}^m(\mathbb{T}^2;\mathbb{R})$, and $\varepsilon > 0$, there exist control parameters $\alpha_1, \ldots, \alpha_4 \in \mathrm{L}^2((0,1);\mathbb{R})$ such that the unique solution*

$$\theta \in \mathrm{C}^0([0,1];\mathrm{H}^m(\mathbb{T}^2;\mathbb{R})) \cap \mathrm{W}^{1,2}((0,1);\mathrm{H}^{m-1}(\mathbb{T}^2;\mathbb{R}))$$

to the linear problem

$$\partial_t \theta + (\overline{\boldsymbol{y}} \cdot \boldsymbol{\nabla})\theta = g \coloneqq \sum_{l=1}^{4} \alpha_l \widetilde{\zeta}_l, \quad \theta(\cdot, 0) = 0 \tag{2.8}$$

obeys

$$\|\theta(\cdot,1) - \theta_1\|_m < \varepsilon \tag{2.9}$$

and the control's space-time average vanishes:

$$\int_0^1 \int_{\mathbb{T}^2} g(\boldsymbol{x}, s)\, \mathrm{d}\boldsymbol{x} \mathrm{d}s = 0. \tag{2.10}$$

Moreover, given a bounded subset $\mathrm{B} \subset \mathrm{H}^m(\mathbb{T}^2;\mathbb{R})$, there exists a continuous linear operator assigning to each $\theta_1 \in \mathrm{B}$ a choice of $\alpha_1, \ldots, \alpha_4 \in \mathrm{L}^2((0,1);\mathbb{R})$ such that the solution θ to (2.8) satisfies $\|\theta(\cdot,1) - \theta_1\|_{m-1} < \varepsilon$.

Proof. By resorting to Theorem 2.5, we select $\overline{g} \in \mathrm{L}^2((0,1);\mathcal{H}_0)$ verifying the controllability problem

$$\partial_t\overline{\theta} + (\overline{\boldsymbol{u}} \cdot \boldsymbol{\nabla})\overline{\theta} = \overline{g}, \quad \overline{\theta}(\cdot,0) = 0, \quad \|\overline{\theta}(\cdot,1) - \theta_1\|_m < \varepsilon.$$

If $\mathrm{B} \subset \mathrm{H}^m(\mathbb{T}^2;\mathbb{R})$ is bounded with $\theta_1 \in \mathrm{B}$, we can assume that $\overline{g}$ is the image of θ_1 under a bounded linear operator C_ε, as provided by Theorem 2.5. Since $\overline{\boldsymbol{y}}$ is given by Theorem 2.1, and the associated flow $\boldsymbol{\mathcal{Y}}$ is, like $\boldsymbol{\mathcal{U}}$, volume preserving, the functions

$$\theta(\boldsymbol{x},t) := \int_0^t g(\boldsymbol{\mathcal{Y}}(\boldsymbol{x},t,s),s)\,\mathrm{d}s, \quad g(\boldsymbol{x},t) := \overline{g}(\boldsymbol{\mathcal{U}}(\boldsymbol{\mathcal{Y}}(\boldsymbol{x},t,1),1,t),t)$$

obey

$$\partial_t\theta + (\overline{\boldsymbol{y}} \cdot \boldsymbol{\nabla})\theta = g, \quad \theta(\cdot,0) = 0, \quad \theta(\cdot,1) = \overline{\theta}(\cdot,1),$$

and it holds

$$\int_{\mathbb{T}^2} (g - \overline{g})(\boldsymbol{x},s)\,\mathrm{d}\boldsymbol{x} = 0$$

for almost all $s \in [0,1]$. □

2.3 Localized temperature controls

The control obtained via Theorem 2.6 is now transformed into one that is physically localized in ω. This new control will be given in terms of 6 fixed profiles that are independent of all data – except the control region ω – imposed in Theorem 1.2. Hereto, let us recall that $\overline{\boldsymbol{y}} = [0, \overline{y}_2]^\top$ from Theorem 2.1 only depends on time and is compactly supported in $(0,1)$.

Theorem 2.7. *There exist profiles $\zeta_1,\ldots,\zeta_6 \in \mathrm{L}^2((0,1);\mathrm{C}^\infty(\mathbb{T}^2;\mathbb{R}))$ that depend only on the control region ω, and for which the following statement holds. Given any $m \in \mathbb{N}$, $\theta_1 \in \mathrm{H}^m$, and $\varepsilon > 0$, there are parameters $\alpha_1,\ldots,\alpha_6 \in \mathrm{L}^2((0,1);\mathbb{R})$ such that the unique solution*

$$\Theta \in \mathrm{C}^0([0,1];\mathrm{H}^m) \cap \mathrm{W}^{1,2}((0,1);\mathrm{H}^{m-1})$$

to the linear problem

$$\partial_t\Theta + (\overline{\boldsymbol{y}} \cdot \boldsymbol{\nabla})\Theta = \mathbb{I}_\omega\eta, \quad \Theta(\cdot,0) = 0, \tag{2.11}$$

where

$$\eta := \sum_{l=1}^{6} \alpha_l\zeta_l \in \mathrm{L}^2((0,1);\mathrm{C}^\infty(\mathbb{T}^2;\mathbb{R})), \quad \int_{\mathbb{T}^2} \eta(\boldsymbol{x},\cdot)\,\mathrm{d}\boldsymbol{x} = 0 \ \text{a.e.}, \tag{2.12}$$

satisfies

$$\|\Theta(\cdot,1) - \theta_1\|_m < \varepsilon. \tag{2.13}$$

In addition, given a bounded subset $\mathrm{B} \subset \mathrm{H}^m$, there exists a continuous linear operator, denoted as $C_\varepsilon \colon \mathrm{H}^m \longrightarrow \mathrm{L}^2((0,1);\mathbb{R})^6$, assigning to each state $\theta_1 \in \mathrm{B}$ parameters $\alpha_1,\ldots,\alpha_6$ such that the solution Θ to (2.6) obeys $\|\Theta(\cdot,1) - \theta_1\|_{m-1} < \varepsilon$.

Proof. Let $\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_4 \in \mathrm{L}^2((0,1);\mathbb{R})$, and the corresponding solution θ to (2.8), be fixed by applying Theorem 2.6 with target state $\theta_1 \in \mathrm{H}^m(\mathbb{T}^2;\mathbb{R})$ such that

$$\|\theta(\cdot,1) - \theta_1\|_m < \varepsilon.$$

In the case that θ_1 is from B, we select $\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_4 \in \mathrm{L}^2((0,1);\mathbb{R})$ as the image of θ_1 under a bounded linear operator, while ensuring $\|\theta(\cdot,1) - \theta_1\|_{m-1} < \varepsilon$.

Step 1. Definition of a localized control. Let us recall from (2.3) the partition of the reference time interval $(0,1)$ with uniform spacing

$$0 < t_c^0 < t_a^1 < t_b^1 < t_c^1 < t_a^2 < t_b^2 < t_c^2 < \cdots < t_a^K < t_b^K < t_c^K < 1.$$

The force $g := \sum_{l=1}^4 \widetilde{\alpha}_l \widetilde{\zeta}_l$, obtained above via Theorem 2.6, is now transformed into a new control f supported in ω. More specifically, we define

$$f(\boldsymbol{x},t) := \chi(x_2) \sum_{k=1}^K \frac{1}{t_b^k - t_a^k} \mathbb{I}_{[t_a^k, t_b^k]}(t) g\left(\boldsymbol{\mathcal{Y}}\left(\boldsymbol{x}, t, \frac{t - t_a^k}{t_b^k - t_a^k}\right), \frac{t - t_a^k}{t_b^k - t_a^k}\right) \tag{2.14}$$

and then demonstrate that the solution $\theta^{\#}$ to

$$\partial_t \theta^{\#} + (\overline{\boldsymbol{y}} \cdot \boldsymbol{\nabla})\theta^{\#} = f, \quad \theta^{\#}(\cdot,0) = 0 \tag{2.15}$$

satisfies $\|\theta^{\#}(\cdot,1) - \theta_1\|_m < \varepsilon$, or $\|\theta^{\#}(\cdot,1) - \theta_1\|_{m-1} < \varepsilon$ if $\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_4$ are chosen to depend continuously on θ_1 from the bounded set B.

Step 2. Checking approximate controllability. Since θ obeys (2.8), and recalling that $\boldsymbol{\mathcal{Y}}$ is the flow associated with $\overline{\boldsymbol{y}}$ from Theorem 2.1, one finds

$$\theta(\boldsymbol{x},1) = \int_0^1 g(\boldsymbol{\mathcal{Y}}(\boldsymbol{x},0,r),r)\,\mathrm{d}r. \tag{2.16}$$

Thus, in view of f's definition in (2.14), the properties of χ and $\boldsymbol{\mathcal{Y}}$ (*cf.* (2.1), (2.2), and Theorem 2.1) imply

$$\begin{aligned}
\theta(\boldsymbol{x},1) &= \sum_{k=1}^K \int_0^1 \chi\left(\boldsymbol{\mathcal{Y}}\left(\boldsymbol{x},0,r(t_b^k - t_a^k) + t_a^k\right)\right) g(\boldsymbol{\mathcal{Y}}(\boldsymbol{x},0,r),r)\,\mathrm{d}r \\
&= \sum_{k=1}^K \frac{1}{t_b^k - t_a^k} \int_0^1 \mathbb{I}_{[t_a^k,t_b^k]}(s)\chi(\boldsymbol{\mathcal{Y}}(\boldsymbol{x},0,s)) g\left(\boldsymbol{\mathcal{Y}}\left(\boldsymbol{x},0,\frac{s-t_a^k}{t_b^k - t_a^k}\right), \frac{s - t_a^k}{t_b^k - t_a^k}\right) \mathrm{d}s \\
&= \int_0^1 f(\boldsymbol{\mathcal{Y}}(\boldsymbol{x},0,s),s)\,\mathrm{d}s,
\end{aligned} \tag{2.17}$$

where we used for $k \in \{1,\ldots,K\}$ the substitutions $r = (s - t_a^k)(t_b^k - t_a^k)^{-1}$. Therefore, the unique solution $\theta^{\#}$ to the problem (2.15) satisfies $\theta^{\#}(\cdot,1) = \theta(\cdot,1)$.

Step 3. Average corrections. In view of the equation for $\theta^{\#}$ in (2.15), and by employing (2.10), (2.16), and (2.17) together with the fact that $\boldsymbol{\mathcal{Y}}$ is volume preserving, it follows that

$$\int_{\mathbb{T}^2} \theta^{\#}(z,t)\,\mathrm{d}z = \int_0^t \int_{\mathbb{T}^2} f(z,s)\,\mathrm{d}z\mathrm{d}s, \quad \int_0^1 \int_{\mathbb{T}^2} f(z,s)\,\mathrm{d}z\,\mathrm{d}s = 0,$$

where f is the function from (2.14). Now, we define

$$\Theta(x_1,x_2,t) \coloneqq \theta^{\#}(x_1,x_2,t) - \frac{\chi(x_2)\int_0^t \int_{\mathbb{T}^2} f(z,s)\,\mathrm{d}z\mathrm{d}s}{\int_{\mathbb{T}^2} \chi(z)\,\mathrm{d}z}.$$

In particular, we have $\Theta(\cdot,0) = \theta^{\#}(\cdot,0)$ and $\Theta(\cdot,1) = \theta^{\#}(\cdot,1)$, and Θ satisfies (2.11) with the control

$$\eta(x_1,x_2,t) \coloneqq f(x_1,x_2,t) - \frac{\overline{y}_2\chi'(x_2)\int_0^t \int_{\mathbb{T}^2} f(z,s)\,\mathrm{d}z\mathrm{d}s - \chi(x_2)\int_{\mathbb{T}^2} f(z,t)\,\mathrm{d}z}{\int_{\mathbb{T}^2} \chi(z)\,\mathrm{d}z} \tag{2.18}$$

of the form (2.12). Moreover, Θ satisfies the controllability condition (2.13), or $\|\Theta(\cdot,1) - \theta_1\|_{m-1} < \varepsilon$ if $\widetilde{\alpha}_1,\dots,\widetilde{\alpha}_4$ are chosen in continuous dependence on θ_1 from the bounded set B.

Step 4. Structure of the control. It remains to name the profiles $\zeta_1,\dots,\zeta_6$ that where implicitly described during the preceding steps. To this end, the function f from (2.14) is expressed by means of

$$\begin{aligned} f(\boldsymbol{x},t) &= \chi(x_2)\sum_{k=1}^{K} \frac{1}{t_b^k - t_a^k}\mathbb{I}_{[t_a^k,t_b^k]}(t)\, g\left(\boldsymbol{\mathcal{Y}}\left(\boldsymbol{x},t,\frac{t-t_a^k}{t_b^k-t_a^k}\right),\frac{t-t_a^k}{t_b^k-t_a^k}\right) \\ &= \sum_{k=1}^{K} \frac{\chi(x_2)\mathbb{I}_{[t_a^k,t_b^k]}(t)}{t_b^k - t_a^k} g\left(\boldsymbol{\mathcal{Y}}(\boldsymbol{x},t,\sigma(t)),\sigma(t)\right), \end{aligned}$$

where

$$\sigma(t) \coloneqq \sum_{l=1}^{K} \mathbb{I}_{[t_a^l,t_b^l]}(t)\frac{t-t_a^l}{t_b^l-t_a^l}. \tag{2.19}$$

Finally, after recalling the definition of $\widetilde{\zeta}_1,\dots,\widetilde{\zeta}_4$ in (2.7), we take $\zeta_1,\dots,\zeta_6$ as any enumeration of the set (in Section 1.2 we denoted $\zeta_1 = \chi$ and $\zeta_2 = \chi'$)

$$\{\chi,\chi'\} \cup \left\{(\boldsymbol{x},t) \mapsto \chi(x_2)\widetilde{\zeta}_i(\boldsymbol{\mathcal{U}}(\boldsymbol{\mathcal{Y}}(\boldsymbol{x},t,1),1,\sigma(t))) \mid i \in \{1,\dots,4\}\right\}. \tag{2.20}$$

The parameters $\alpha_1,\dots,\alpha_6 \in \mathrm{L}^2((0,1);\mathbb{R})$ are then determined from the above choice of $\widetilde{\alpha}_1,\dots,\widetilde{\alpha}_4$ and the definition of η in (2.18). When the target θ_1 varies in a bounded set of H^m, the formulas (2.14) and (2.18) allow taking $\widetilde{\alpha}_1,\dots,\widetilde{\alpha}_6$ as the image of θ_1 under a bounded linear operator. Because χ, $\boldsymbol{\mathcal{Y}}$, $\boldsymbol{\mathcal{U}}$, and σ are universal objects for fixed ω, the set in (2.20) is not affected by the choice of the initial and target states, the viscosity, the thermal diffusivity, the external forces, and also not by the approximation error specified in Theorems 1.2 and 1.3.

□

3 Proofs of the main results

To avoid for simplicity the discussion of weak notions of solutions, we assume without loss of generality that $r \geq 2$ in Theorems 1.2 and 1.3. Indeed, if it would hold $r < 2$, and given that $(\boldsymbol{\Phi}_{\mathrm{ext}}, \psi_{\mathrm{ext}}) \in \mathrm{L}^2((0,T);\mathbf{H}^2 \times \mathrm{H}^2)$, the corresponding uncontrolled trajectory naturally regularizes due to the known parabolic smoothing effects, and after any short time assumes states in $\mathbf{H}^2 \times \mathrm{H}^2$, which can be taken as the new initial data.

Owing to classical elliptic regularity estimates (*cf.* (3.4) below), for proving Theorems 1.2 and 1.3 it suffices to determine control forces $\eta, \overline{\eta} \in \mathrm{L}^2((0,T);\mathrm{C}^\infty(\mathbb{T}^2;\mathbb{R}))$ of the type (1.5) that ensure an estimate of the form

$$\|\boldsymbol{\nabla} \wedge \boldsymbol{u}(\cdot,T) - \boldsymbol{\nabla} \wedge \boldsymbol{u}_T\|_{r-1} + \|\theta(\cdot,T) - \theta_T\|_r + \left|\int_{\mathbb{T}^2} \boldsymbol{u}(\boldsymbol{x},T)\,\mathrm{d}\boldsymbol{x}\right| < \varepsilon, \tag{3.1}$$

where $\boldsymbol{\nabla} \wedge \boldsymbol{u} \coloneqq \partial_1 u_2 - \partial_2 u_1$ is the curl of $\boldsymbol{u}$ and $\varepsilon > 0$ is the approximation accuracy selected in Theorems 1.2 and 1.3.

Vorticity-temperature formulation. For initial data $(\boldsymbol{u}_0, \theta_0) \in \mathbf{H}^r \times \mathrm{H}^r$ and prescribed forces $(\boldsymbol{\Phi}_{\mathrm{ext}}, \psi_{\mathrm{ext}}) \in \mathrm{L}^2((0,T);\mathbf{H}^r \times \mathrm{H}^r)$, let $(\boldsymbol{u}, \theta, p)$ be the solution to (1.1), and denote the vorticity $w = \boldsymbol{\nabla} \wedge \boldsymbol{u}$. Then, the triple $(\boldsymbol{u}, w, \theta)$ satisfies in $\mathbb{T}^2 \times (0,T)$ the problem

$$\begin{gathered}\partial_t w - \nu\Delta w + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\, w = \partial_1\theta + \varphi_{\mathrm{ext}}, \quad \partial_t\theta - \tau\Delta\theta + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\theta = \mathbb{I}_\omega \eta + \psi_{\mathrm{ext}},\\ \boldsymbol{\nabla}\wedge\boldsymbol{u} = w, \quad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \quad w(\cdot,0) = w_0, \quad \theta(\cdot,0) = \theta_0,\end{gathered} \tag{3.2}$$

where $w_0 = \boldsymbol{\nabla}\wedge\boldsymbol{u}_0$ and $\varphi_{\mathrm{ext}} = \boldsymbol{\nabla}\wedge\boldsymbol{\Phi}_{\mathrm{ext}}$. Vice versa, if $(\boldsymbol{u}, w, \theta)$ solves (3.2) and obeys $\int_{\mathbb{T}^2} u_2(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x} = \int_0^t \int_{\mathbb{T}^2} \theta(\boldsymbol{x},s)\,\mathrm{d}\boldsymbol{x}\mathrm{d}s$ for all $t \in [0,T]$, one can recover the pressure p, uniquely up to an additive constant depending on time, such that $(\boldsymbol{u},\theta,p)$ is a solution to (1.1); *e.g.*, see [22].

Inverting the curl operator. Given any $m \in \mathbb{N}$, let $\boldsymbol{\Upsilon}\colon \mathrm{H}^{m-1}\times\mathbb{R}^2 \longrightarrow \mathbf{V}^m$ be the following solenoidal realization of $(\boldsymbol{\nabla}\wedge\,)^{-1}$: for elements $z \in \mathrm{H}^{m-1}$ and $\boldsymbol{A} \in \mathbb{R}^2$, the vector field $\boldsymbol{\Upsilon}(z,\boldsymbol{A}) \in \mathbf{V}^m$ is defined as the unique solution to the planar div-curl problem

$$\boldsymbol{\nabla}\wedge\boldsymbol{\Upsilon}(z,\boldsymbol{A}) = z, \quad \boldsymbol{\nabla}\cdot\boldsymbol{\Upsilon}(z,\boldsymbol{A}) = 0, \quad \int_{\mathbb{T}^2} \boldsymbol{\Upsilon}(z,\boldsymbol{A})(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \boldsymbol{A}. \tag{3.3}$$

In fact, one has the representation

$$\boldsymbol{\Upsilon}(z,\boldsymbol{A}) = \boldsymbol{\nabla}^\perp\psi + \boldsymbol{A}, \quad \boldsymbol{\nabla}^\perp\psi \coloneqq [\partial_2\psi, -\partial_1\psi]^\top,$$

where the stream function ψ solves Poisson's equation $\Delta\psi = -z$. Then, by the elliptic theory for the Laplacian, there exists a constant $C_0 > 0$ such that

$$\|\boldsymbol{\Upsilon}(z,\boldsymbol{A})\|_m \leq C_0(\|z\|_{m-1} + |\boldsymbol{A}|) \tag{3.4}$$

for all $z \in \mathrm{H}^{m-1}$ and $\boldsymbol{A} \in \mathbb{R}^2$.

3.1 Well-posedness and hydrodynamic scaling

We recall that, like the Navier–Stokes system in 2D, the two-dimensional Boussinesq system is globally well-posed in the space

$$\mathcal{X}_T^m := \mathcal{A}_T^{m-1} \times \mathcal{A}_T^m, \quad \|(f,g)\|_{\mathcal{X}_T^m} := \|f\|_{\mathcal{A}_T^{m-1}} + \|g\|_{\mathcal{A}_T^m},$$

where $m \in \mathbb{N}$ and $\mathcal{A}_T^m := \mathrm{C}^0([0,T];\mathrm{H}^m(\mathbb{T}^2;\mathbb{R})) \cap \mathrm{L}^2((0,T);\mathrm{H}^{m+1}(\mathbb{T}^2;\mathbb{R}))$ is endowed with $\|\cdot\|_{\mathcal{A}_T^m} := \|\cdot\|_{\mathrm{C}^0([0,T];\mathrm{H}^m(\mathbb{T}^2;\mathbb{R}))} + \|\cdot\|_{\mathrm{L}^2((0,T);\mathrm{H}^{m+1}(\mathbb{T}^2;\mathbb{R}))}$.

The following well-posedness result can be shown by analysis similar to the incompressible Navier–Stokes system; *e.g.*, see [22].

Proposition 3.1. *Given arbitrary initial states* $(w_0, \theta_0) \in \mathrm{H}^{m-1} \times \mathrm{H}^m(\mathbb{T}^2;\mathbb{R})$*, external forces* $(h_1, h_2) \in \mathrm{L}^2((0,T);\mathrm{H}^{m-2} \times \mathrm{H}^{m-1}(\mathbb{T}^2;\mathbb{R}))$*, and average* $\boldsymbol{A} \in \mathrm{W}^{1,2}((0,T);\mathbb{R}^2)$*, there exists a unique solution* $(w, \theta) \in \mathcal{X}_T^m$ *to the Boussinesq system in vorticity-temperature form*

$$\begin{aligned} \partial_t w - \nu \Delta w + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\, w = \partial_1 \theta + h_1, \quad \partial_t \theta - \tau \Delta \theta + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\, \theta = h_2, \\ \boldsymbol{u}(\cdot, t) = \boldsymbol{\Upsilon}(w, \boldsymbol{A}), \quad w(\cdot, 0) = w_0, \quad \theta(\cdot, 0) = \theta_0. \end{aligned} \tag{3.5}$$

The resolving operator S_T *associated with* (3.5) *is the mapping*

$$\begin{gathered} \mathrm{H}^{m-1} \times \mathrm{H}^m(\mathbb{T}^2;\mathbb{R}) \times \mathrm{L}^2((0,T);\mathrm{H}^{m-2} \times \mathrm{H}^{m-1}(\mathbb{T}^2;\mathbb{R})) \times \mathrm{W}^{1,2}((0,T);\mathbb{R}^2) \longrightarrow \mathcal{X}_T^m, \\ (w_0, \theta_0, h_1, h_2, \boldsymbol{A}) \longmapsto S_T(w_0, \theta_0, h_1, h_2, \boldsymbol{A}) := (w, \theta). \end{gathered}$$

The next result relates the solutions to (3.2) at a small time with the solutions to linear transport problems with drift $\overline{\boldsymbol{y}}$ at time $t = 1$. This approach is inspired by [7] and the present version particularly builds on the recent works [19, 20].

Theorem 3.2. *Given* $T > 0$*,* $m \geq 2$*, let initial states* $(w_0, \theta_0) \in \mathrm{H}^m \times \mathrm{H}^{m+1}$ *and external forces* $(\varphi_{\mathrm{ext}}, \psi_{\mathrm{ext}}) \in \mathrm{L}^2((0,T);\mathrm{H}^{m-2} \times \mathrm{H}^{m-1})$ *be arbitrarily fixed. Moreover, denote by* $(v_\delta, \vartheta_\delta)_{\delta \in (0,1)}$ *the solution family to the linear transport problems*

$$\begin{aligned} \partial_t v_\delta + (\overline{\boldsymbol{y}} \cdot \boldsymbol{\nabla}) v_\delta = \partial_1 \vartheta_\delta, \quad \partial_t \vartheta_\delta + (\overline{\boldsymbol{y}} \cdot \boldsymbol{\nabla}) \vartheta_\delta = \eta_\delta, \\ v_\delta(\cdot, 0) = w_0, \quad \vartheta_\delta(\cdot, 0) = \delta \theta_0, \end{aligned} \tag{3.6}$$

where $\overline{\boldsymbol{y}}$ *is the vector field from Theorem 2.1 and* $\eta_\delta \subset \mathrm{L}^2((0,1);\mathrm{C}^\infty(\mathbb{T}^2;\mathbb{R}))$ *is chosen in a way that*

$$\int_{\mathbb{T}^2} \eta_\delta(\boldsymbol{x}, \cdot)\, \mathrm{d}\boldsymbol{x} = 0 \ \text{a.e.}, \qquad \sup_{t \in [0,1]} \|\vartheta_\delta(\cdot, t)\|_{m+1} = \mathcal{O}(\delta) \text{ as } \delta \longrightarrow 0. \tag{3.7}$$

Then, one has the convergence

$$\lim_{\delta \to 0} \|S_\delta(w_0, \theta_0, \varphi_{\mathrm{ext}}, \psi_{\mathrm{ext}} + H_\delta, \overline{\boldsymbol{y}}_\delta)|_{t=\delta} - (v_\delta, \delta^{-1}\vartheta_\delta)(\cdot, 1)\|_{\mathrm{H}^{m-1} \times \mathrm{H}^m(\mathbb{T}^2;\mathbb{R})} = 0,$$

where

$$H_\delta(\cdot, t) := \delta^{-2} \eta_\delta(\cdot, \delta^{-1} t), \quad \overline{\boldsymbol{y}}_\delta(t) := \delta^{-1} \overline{\boldsymbol{y}}(\delta^{-1} t),$$

uniformly with respect to $(\varphi_{\mathrm{ext}}, \psi_{\mathrm{ext}})$ *from bounded subsets of* $\mathrm{L}^2((0,T);\mathrm{H}^{m-2} \times \mathrm{H}^{m-1})$.

Proof. For any $\delta \in (0,1)$, let $(w,\theta) \in \mathcal{X}^m_\delta$ be the solution to the nonlinear problem (3.2) driven by H_δ, and with velocity average $\overline{\boldsymbol{y}}_\delta$. Namely, we take $(w,\theta) = S_\delta(w_0, \theta_0, \varphi_{\mathrm{ext}}, \psi_{\mathrm{ext}} + H_\delta, \overline{\boldsymbol{y}}_\delta)$, with associated velocity $\boldsymbol{u} = \boldsymbol{\Upsilon}\left(w, \overline{\boldsymbol{y}}_\delta\right)$, where the div-curl solution operator $\boldsymbol{\Upsilon}$ is defined via (3.3). Then, we make an ansatz of the form

$$w = z_\delta + r, \quad \boldsymbol{u} = \overline{\boldsymbol{y}}_\delta + \boldsymbol{Z}_\delta + \boldsymbol{R}, \quad \theta = \theta_\delta + s, \tag{3.8}$$

where

$$z_\delta(\cdot, t) \coloneqq v_\delta(\cdot, \delta^{-1}t), \quad \theta_\delta(\cdot,t) \coloneqq \delta^{-1}\vartheta_\delta(\cdot, \delta^{-1}t), \quad \boldsymbol{Z}_\delta \coloneqq \boldsymbol{\Upsilon}(z_\delta, \boldsymbol{0}), \quad \boldsymbol{R} \coloneqq \boldsymbol{\Upsilon}(r, \boldsymbol{0}).$$

The theorem will be proved by showing that

$$\|r(\cdot,\delta)\|_{m-1} + \|s(\cdot,\delta)\|_m \longrightarrow 0 \text{ as } \delta \longrightarrow 0, \tag{3.9}$$

uniformly for $(\varphi_{\mathrm{ext}}, \psi_{\mathrm{ext}})$ from bounded subsets of $\mathrm{L}^2((0,1); \mathrm{H}^{m-2} \times \mathrm{H}^{m-1})$.

Step 1. Description of remainders. By plugging the ansatz (3.8) into the equation (3.2) satisfied by (w,θ), one finds that r and s solve the evolutionary system

$$\begin{aligned} \partial_t r - \nu\Delta r + \left((\overline{\boldsymbol{y}}_\delta + \boldsymbol{Z}_\delta + \boldsymbol{R}) \cdot \boldsymbol{\nabla}\right) r + (\boldsymbol{R} \cdot \boldsymbol{\nabla}) z_\delta = \Xi_\delta + \partial_1 s, \\ \partial_t s - \tau\Delta s + \left((\overline{\boldsymbol{y}}_\delta + \boldsymbol{Z}_\delta + \boldsymbol{R}) \cdot \boldsymbol{\nabla}\right) s + (\boldsymbol{R} \cdot \boldsymbol{\nabla})\theta_\delta = \Lambda_\delta \end{aligned} \tag{3.10}$$

with initial conditions $r(\cdot,0) = s(\cdot,0) = 0$, and where

$$\Xi_\delta \coloneqq \varphi_{\mathrm{ext}} - (\boldsymbol{Z}_\delta \cdot \boldsymbol{\nabla}) z_\delta + \nu\Delta z_\delta, \quad \Lambda_\delta \coloneqq \psi_{\mathrm{ext}} - (\boldsymbol{Z}_\delta \cdot \boldsymbol{\nabla})\theta_\delta + \tau\Delta\theta_\delta.$$

Moreover, one has the elliptic estimates (*cf.* (3.4))

$$\|\boldsymbol{R}(\cdot,t)\|_m \leq C_0\|r(\cdot,t)\|_{m-1}, \quad \|\boldsymbol{Z}_\delta(\cdot,t)\|_m \leq C_0\|z_\delta(\cdot,t)\|_{m-1}, \quad t \in [0,\delta]. \tag{3.11}$$

Step 2. *A priori* estimates. Since the vorticity-temperature coupling in (3.2) is linear, and due to the dissipation of both w and θ, the subsequent estimates are similar to those provided in a related context for the Navier–Stokes system by [20, Proof of Lemma 5.5] and [19, Proposition 2.2]. That is, after formally multiplying in (3.10) with $(-\Delta)^{m-1}r$ and $(-\Delta)^m s$ respectively, integration by parts, (3.11), and known inequalities yield

$$\begin{aligned} &\frac{1}{2}\|r(\cdot,t)\|^2_{m-1} + \frac{1}{2}\|s(\cdot,t)\|^2_m + \nu\int_0^t \|r(\cdot,\sigma)\|^2_m \,\mathrm{d}\sigma + \tau\int_0^t \|s(\cdot,\sigma)\|^2_{m+1}\,\mathrm{d}\sigma \\ &\leq \int_0^t \left(\|\Xi_\delta(\cdot,\sigma)\|_{m-2}\|r(\cdot,\sigma)\|_m \,\mathrm{d}\sigma + \|\Lambda_\delta(\cdot,\sigma)\|_{m-1}\|s(\cdot,\sigma)\|_{m+1}\right)\mathrm{d}\sigma \\ &\quad + \int_0^t \|\boldsymbol{R}(\cdot,\sigma)\|_m \left(\|z_\delta(\cdot,\sigma)\|_m\|r(\cdot,\sigma)\|_{m-1} + \|r(\cdot,\sigma)\|_{m-1}\|r(\cdot,\sigma)\|_m\right)\mathrm{d}\sigma \\ &\quad + \int_0^t \|\boldsymbol{R}(\cdot,\sigma)\|_m \left(\|\theta_\delta(\cdot,\sigma)\|_{m+1}\|s(\cdot,\sigma)\|_m + \|s(\cdot,\sigma)\|_m\|s(\cdot,\sigma)\|_{m+1}\right)\mathrm{d}\sigma \\ &\quad + \int_0^t \left(\|\overline{\boldsymbol{y}}_\delta(\sigma) + \boldsymbol{Z}_\delta(\cdot,\sigma)\|_{m+1}\|r(\cdot,\sigma)\|^2_{m-1} + \|s(\cdot,\sigma)\|_m\|r(\cdot,\sigma)\|_{m-1}\right)\mathrm{d}\sigma \\ &\quad + \int_0^t \|\overline{\boldsymbol{y}}_\delta(\sigma) + \boldsymbol{Z}_\delta(\cdot,\sigma)\|_{m+1}\|s(\cdot,\sigma)\|^2_m \,\mathrm{d}\sigma. \end{aligned}$$

In order to further estimate the right-hand side of the previous inequality, we again use (3.11), while also accounting for the δ-scaling by substituting $\sigma \leftrightarrow \delta\sigma$ under several of the integral signs. *E.g.*, for $t \in (0,\delta)$, it follows that

$$\int_0^t \|f(\cdot,\sigma)\|_l \,\mathrm{d}\sigma \leq \min\left\{\delta \int_0^1 \|f(\cdot,\delta\sigma)\|_l \,\mathrm{d}\sigma, \int_0^\delta \|f(\cdot,\sigma)\|_l \,\mathrm{d}\sigma\right\} \tag{3.12}$$

for all $f \in \mathrm{L}^1((0,\delta);\mathrm{H}^l(\mathbb{T}^2;\mathbb{R}))$ with $l \geq 0$. The relations in (3.12), combined with the respective boundedness of Ξ_δ in $\mathrm{L}^1((0,\delta);\mathrm{H}^{m-2}(\mathbb{T}^2;\mathbb{R}))$ and of $\overline{\boldsymbol{y}}$ in $\mathrm{C}^0([0,1];\mathbb{R}^2)$, yield $\lim\limits_{\delta\to 0} \int_0^\delta \|\Xi_\delta(\cdot,\sigma)\|_{m-2} \,\mathrm{d}\sigma = 0$ and

$$\lim_{\delta\to 0} \int_0^\delta \|\overline{\boldsymbol{y}}_\delta(\sigma) + \boldsymbol{Z}_\delta(\cdot,\sigma)\|_{m+1} \,\mathrm{d}\sigma \leq \sup_{s\in[0,1]} |\overline{\boldsymbol{y}}(s)|.$$

In particular, thanks to the assumptions in (3.7), one can infer

$$\int_0^\delta \|\theta_\delta(\cdot,\sigma)\|_{m+1}^2 \,\mathrm{d}\sigma \leq \delta^{-1} \sup_{s\in[0,1]} \|\vartheta_\delta(\cdot,s)\|_{m+1}^2 = \mathscr{O}(\delta) \text{ as } \delta \longrightarrow 0,$$
$$\lim_{\delta\to 0} \int_0^\delta \|\Lambda_\delta(\cdot,\sigma)\|_{m-1} \,\mathrm{d}\sigma = 0.$$

Therefore, in view of Grönwall's lemma, and by essentially copying the analysis from [20, Proof of Lemma 5.5], it follows that there is a constant $C > 0$, which is independent of $\delta \in (0,1)$, $t \in [0,\delta]$, and (w_0,θ_0) varying in a bounded subset of $\mathrm{H}^m \times \mathrm{H}^{m+1}$, such that

$$\|r(\cdot,t)\|_{m-1}^2 + \|s(\cdot,t)\|_m^2 \leq C_\delta + C\int_0^t \left(\|r(\cdot,\sigma)\|_{m-1}^4 + \|s(\cdot,\sigma)\|_m^4\right) \mathrm{d}\sigma,$$

where the family of constants $(C_\delta)_{\delta\in(0,1)}$ satisfies $\lim_{\delta\to 0} C_\delta = 0$. Finally, after denoting

$$\Psi(t) \coloneqq C_\delta + C\int_0^t \left(\|r(\cdot,\sigma)\|_{m-1}^4 + \|s(\cdot,\sigma)\|_m^4\right) \mathrm{d}\sigma,$$

the convergence asserted in (3.9) follows by utilizing that Ψ obeys the differential inequality $\frac{\mathrm{d}}{\mathrm{d}t}\Psi \leq C\Psi^2$; for situations of similar nature, see, *e.g.*, [19, Proposition 2.2] and [20, Proof of Lemma 5.5]. □

3.2 Controllability of the vorticity-temperature formulation

Let $\zeta_1,\ldots,\zeta_6 \in \mathrm{L}^2((0,1);\mathrm{C}^\infty(\mathbb{T}^2;\mathbb{R}))$ be the profiles introduced in Section 1.2 and fixed via (2.20). Moreover, we recall that $\overline{\boldsymbol{y}}$ is obtained via Theorem 2.1 and continue using the notation $\overline{\boldsymbol{y}}_\delta(t) = \delta^{-1}\overline{\boldsymbol{y}}(\delta^{-1}t)$ for $\delta \in (0,1)$ and $t \in [0,\delta]$.

The following result, which is a consequence of Theorems 2.7 and 3.2, allows to steer the temperature towards any state in a small time, while keeping the vorticity close to the initial one. The control is hereby finitely-decomposable and physically localized.

Theorem 3.3. *For any given $T > 0$, $m \geq 2$, $(w_0, \theta_0, \theta_1) \in \mathrm{H}^m \times \mathrm{H}^{m+1} \times \mathrm{H}^{m+1}$, and $(\varphi_{\mathrm{ext}}, \psi_{\mathrm{ext}}) \in \mathrm{L}^2((0,T); \mathrm{H}^{m-2} \times \mathrm{H}^{m-1})$, there exist parameters*

$$(\alpha_{\delta,1}, \dots, \alpha_{\delta,6})_{\delta \in (0,1)} \subset \mathrm{L}^2((0,1); \mathbb{R}), \quad \sum_{l=1}^{6} \int_{\mathbb{T}^2} \alpha_{\delta,l} \zeta_l(\boldsymbol{x}, \cdot) \, \mathrm{d}\boldsymbol{x} = 0 \;\; a.e. \tag{3.13}$$

such that, when $\delta \longrightarrow 0$, one has in $\mathrm{H}^{m-1} \times \mathrm{H}^m$ the convergence

$$S_\delta \left(w_0, \theta_0, \varphi_{\mathrm{ext}}, \psi_{\mathrm{ext}} + \delta^{-2} \sum_{l=1}^{6} \alpha_{\delta,l}(\delta^{-1} \cdot) \zeta_l(\cdot, \delta^{-1} \cdot), \overline{\boldsymbol{y}}_\delta \right)|_{t=\delta} \longrightarrow (w_0, \theta_1),$$

uniformly for $(\varphi_{\mathrm{ext}}, \psi_{\mathrm{ext}})$ from bounded subsets of $\mathrm{L}^2((0,T); \mathrm{H}^{m-2} \times \mathrm{H}^{m-1})$.

Proof. Let $\rho \in (0,1)$ and take $(\widetilde{v}_\rho, \widetilde{\vartheta}_\rho)$ as the unique solution to the homogeneous linear problem

$$\partial_t \widetilde{v}_\rho + (\overline{\boldsymbol{y}} \cdot \boldsymbol{\nabla}) \widetilde{v}_\rho = \partial_1 \widetilde{\vartheta}_\rho, \quad \partial_t \widetilde{\vartheta}_\rho + (\overline{\boldsymbol{y}} \cdot \boldsymbol{\nabla}) \widetilde{\vartheta}_\rho = 0, \quad (\widetilde{v}_\rho, \widetilde{\vartheta}_\rho)(\cdot, 0) = (w_0, \rho\theta_0). \tag{3.14}$$

By the properties of $\boldsymbol{\mathcal{Y}}$ from Theorem 2.1, it follows that

$$\widetilde{\vartheta}_\rho(\cdot, 1) = \rho\theta_0, \quad \widetilde{v}_\rho(\cdot, 1) = w_0 + \widetilde{v}_{\rho,1}, \quad \widetilde{v}_{\rho,1} := \int_0^1 \partial_1 \widetilde{\vartheta}_\rho(\boldsymbol{\mathcal{Y}}(\cdot, 1, s), s) \, \mathrm{d}s.$$

In addition, basic estimates yield

$$\sup_{t \in [0,1]} \|\widetilde{\vartheta}_\rho(\cdot, t)\|_{m+1} + \|\widetilde{v}_{\rho,1}\|_m = \mathcal{O}(\rho) \text{ as } \rho \longrightarrow 0. \tag{3.15}$$

Now, let any $\varepsilon > 0$ be fixed. Then, for each $\rho \in (0,1)$, an application of Theorem 2.7 with the target $\Theta_1 := \rho(\theta_1 - \theta_0)$ provides $(\alpha_{\rho,1}, \dots, \alpha_{\rho,6})_{\rho \in (0,1)} \subset \mathrm{L}^2((0,1); \mathbb{R})$ such that the respective solution

$$(V_\rho, \Theta_\rho) \in \mathrm{C}^0([0,1]; \mathrm{H}^m \times \mathrm{H}^{m+1}) \cap \mathrm{W}^{1,2}((0,1); \mathrm{H}^{m-1} \times \mathrm{H}^m)$$

to

$$\partial_t V_\rho + (\overline{\boldsymbol{y}} \cdot \boldsymbol{\nabla}) V_\rho = \partial_1 \Theta_\rho, \quad \partial_t \Theta_\rho + (\overline{\boldsymbol{y}} \cdot \boldsymbol{\nabla}) \Theta_\rho = \sum_{l=1}^{6} \alpha_{\rho,l} \zeta_l, \quad V_\rho(\cdot, 0) = \Theta_\rho(\cdot, 0) = 0, \tag{3.16}$$

obeys

$$\|\Theta_\rho(\cdot, 1) - \Theta_1\|_m < \varepsilon.$$

Moreover, the control parameters are chosen such that

$$\begin{gathered} \|\sum_{l=1}^{6} \alpha_{\rho,l} \zeta_l\|_{\mathrm{L}^2((0,1);\mathrm{H}^{m+1})} \leq \rho c_\varepsilon \left(\sum_{l=1}^{6} \|\zeta_l\|_{\mathrm{L}^2((0,1);\mathrm{H}^{m+1})} \right) \|\theta_1 - \theta_0\|_{m+1}, \\ \sum_{l=1}^{6} \int_{\mathbb{T}^2} \alpha_{\rho,l} \zeta_l(\boldsymbol{x}, \cdot) \, \mathrm{d}\boldsymbol{x} = 0 \text{ a.e.}, \end{gathered} \tag{3.17}$$

where c_ε is the operator norm of the bounded linear operator C_ε from Theorem 2.7. Further, by estimating V_ρ and Θ_ρ in (3.16) using (3.17), one finds

$$\sup_{t\in[0,1]} \|\Theta_\rho(\cdot,t)\|_{m+1} + \|w_0 - \widetilde{v}_\rho(\cdot,1)\|_m + \|V_\rho(\cdot,1)\|_m = \mathcal{O}(\rho) \text{ as } \rho \longrightarrow 0. \quad (3.18)$$

Now, the pair $(v_\rho, \vartheta_\rho) := (\widetilde{v}_\rho + V_\rho, \widetilde{\vartheta}_\rho + \Theta_\rho)$ solves the linear reference system (3.6) in Theorem 3.2 with force $\eta_\rho = \sum_{l=1}^{6} \alpha_{\rho,l}\zeta_l$, and it holds

$$\|\vartheta_\rho(\cdot,1) - \rho\theta_1\|_m < \varepsilon.$$

Moreover, from (3.15) and (3.18) it follows that

$$\sup_{t\in[0,1]} \|\vartheta_\rho(\cdot,t)\|_{m+1} + \|v_\rho(\cdot,1) - w_0\|_{m-1} = \mathcal{O}(\rho) \text{ as } \rho \longrightarrow 0.$$

Hence, by Theorem 3.2, there exists $\delta_* > 0$ such that

$$\|S_\delta\left(w_0, \theta_0, \varphi_{\mathrm{ext}}, \psi_{\mathrm{ext}} + \delta^{-2}\eta_\delta(\cdot, \delta^{-1}\cdot), \overline{\boldsymbol{y}}_\delta\right)|_{t=\delta} - (w_0, \theta_1)\|_{\mathrm{H}^{m-1}\times\mathrm{H}^m} < \varepsilon$$

for all $\delta \in (0, \delta_*)$. □

The following result specifies types of target states that are approximately reached naturally (without using an additive control) when starting from appropriate initial data. To this end, let $\Pi_1 \colon \mathcal{A}^m_T \times \mathcal{A}^{m+1}_T \longrightarrow \mathcal{A}^m_T$ denote the projection to the first component.

Theorem 3.4. *Let* $m \geq 2, T > 0$, *average-free* $\widetilde{\xi}, \widehat{\xi} \in \mathrm{C}^\infty(\mathbb{T}^2;\mathbb{R})$, $(w_0, \theta_0) \in \mathrm{H}^m \times \mathrm{H}^{m+1}$, *and* $(\varphi_{\mathrm{ext}}, \psi_{\mathrm{ext}}) \in \mathrm{L}^2((0,T);\mathrm{H}^{m-2}\times\mathrm{H}^{m-1})$. *As* $\delta \longrightarrow 0$, *one has*

$$\Pi_1 S_\delta(w_0, \theta_0 - \delta^{-1}\widetilde{\xi}, \varphi_{\mathrm{ext}}, \psi_{\mathrm{ext}}, \mathbf{0})|_{t=\delta} \longrightarrow w_0 - \partial_1\widetilde{\xi}, \quad (3.19)$$

$$S_\delta(w_0 + \delta^{-1/2}\widehat{\xi}, \theta_0, \varphi_{\mathrm{ext}}, \psi_{\mathrm{ext}}, \mathbf{0})|_{t=\delta} - (0, \delta^{-1/2}\widehat{\xi}) \longrightarrow (w_0 - (\boldsymbol{\Upsilon}(\widehat{\xi}, \mathbf{0})\cdot\boldsymbol{\nabla})\widehat{\xi}, \theta_0), \quad (3.20)$$

in the norms of H^{m-1} *respectively* $\mathrm{H}^{m-1}\times\mathrm{H}^m$. *Both convergences in* (3.19) *and* (3.20) *are uniform with respect to* $(\varphi_{\mathrm{ext}}, \psi_{\mathrm{ext}})$ *from bounded subsets of* $\mathrm{L}^2((0,T);\mathrm{H}^{m-2}\times\mathrm{H}^{m-1})$.

Proof. We develop a simplified version of the arguments given in [4, Proof of Proposition 1.2] for the 3D primitive equations. First, we observe that

$$(W_\delta, \Theta_\delta) = S_\delta(w_0, \theta_0 - \delta^{-1}\widetilde{\xi}, \varphi_{\mathrm{ext}}, \psi_{\mathrm{ext}}, \mathbf{0}) + (0, \delta^{-1}\widetilde{\xi})$$

solves

$$\begin{aligned}
&\partial_t W_\delta - \nu\Delta W_\delta + (\boldsymbol{U}_\delta\cdot\boldsymbol{\nabla})\, W_\delta = \partial_1(\Theta_\delta - \delta^{-1}\widetilde{\xi}) + \varphi_{\mathrm{ext}},\\
&\partial_t\Theta_\delta - \tau\Delta(\Theta_\delta - \delta^{-1}\widetilde{\xi}) + (\boldsymbol{U}_\delta\cdot\boldsymbol{\nabla})(\Theta_\delta - \delta^{-1}\widetilde{\xi}) = \psi_{\mathrm{ext}},\\
&\boldsymbol{\nabla}\wedge\boldsymbol{U}_\delta = W_\delta, \quad \boldsymbol{\nabla}\cdot\boldsymbol{U}_\delta = 0, \quad \int_{\mathbb{T}^2}\boldsymbol{U}_\delta(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x} = \mathbf{0},\\
&W_\delta(\cdot,0) = W_0 := w_0, \quad \Theta_\delta(\cdot,0) = \Theta_0 := \theta_0.
\end{aligned} \quad (3.21)$$

Showing (3.19)**.** It remains to verify for the solutions $(W_\delta, \boldsymbol{U}_\delta, \Theta_\delta)_{\delta\in(0,1)}$ to (3.21) the convergence

$$\lim_{\delta\to 0} \|W_\delta(\cdot,\delta) - (W_0 - \partial_1\widetilde{\xi})\|_{m-1} = 0. \tag{3.22}$$

Establishing (3.22) means showing $\|Q_\delta(\cdot,\delta)\|_{m-1} \longrightarrow 0$ as $\delta \longrightarrow 0$ for the functions

$$Q_\delta(\boldsymbol{x},t) \coloneqq W_\delta(\boldsymbol{x},t) - W_0(\boldsymbol{x}) + \delta^{-1}t\partial_1\widetilde{\xi}(\boldsymbol{x}), \quad \delta \in (0,1).$$

In fact, each Q_δ obeys the initial condition $Q_\delta(\cdot,0) = 0$ and solves

$$\begin{aligned}\partial_t Q_\delta - \nu\Delta Q_\delta + (\boldsymbol{V}_\delta\cdot\boldsymbol{\nabla})Q_\delta &= \varphi_{\mathrm{ext}} + \partial_1(\Theta_\delta - R_\delta) + \partial_1 R_\delta - \nu\Delta(Q_\delta - W_\delta)\\ +((\boldsymbol{U}_\delta - \boldsymbol{V}_\delta)\cdot\boldsymbol{\nabla})(Q_\delta - W_\delta) &+ (\boldsymbol{V}_\delta\cdot\boldsymbol{\nabla})(Q_\delta - W_\delta) + ((\boldsymbol{V}_\delta - \boldsymbol{U}_\delta)\cdot\boldsymbol{\nabla})Q_\delta,\end{aligned} \tag{3.23}$$

where $\boldsymbol{V}_\delta = \boldsymbol{\Upsilon}(Q_\delta, \boldsymbol{0})$ and

$$R_\delta(\cdot,t) \coloneqq \Theta_\delta(\cdot,t) - \Theta_0 + \delta^{-1}t\tau\Delta\widetilde{\xi} - \delta^{-1}t\left(\boldsymbol{\Upsilon}\left(W_0 - \frac{\delta^{-1}t\partial_1\widetilde{\xi}}{2}, \boldsymbol{0}\right)\cdot\boldsymbol{\nabla}\right)\widetilde{\xi} \tag{3.24}$$

for $\delta \in (0,1)$ and $t \in [0,\delta]$. In particular, one has $R_\delta(\cdot,0) = 0$, and R_δ satisfies

$$\begin{aligned}\partial_t R_\delta - \tau\Delta R_\delta + (\boldsymbol{V}_\delta\cdot\boldsymbol{\nabla})R_\delta &= \psi_{\mathrm{ext}} - \tau\Delta(R_\delta - \Theta_\delta) + (\boldsymbol{V}_\delta\cdot\boldsymbol{\nabla})(R_\delta - \Theta_\delta)\\ +((\boldsymbol{V}_\delta - \boldsymbol{U}_\delta)\cdot\boldsymbol{\nabla})R_\delta &+ ((\boldsymbol{V}_\delta - \boldsymbol{U}_\delta)\cdot\boldsymbol{\nabla})(\Theta_\delta - R_\delta) + \delta^{-1}(\boldsymbol{V}_\delta\cdot\boldsymbol{\nabla})\widetilde{\xi}.\end{aligned} \tag{3.25}$$

Now, given any $a \geq 1$, it follows that

$$\|(Q_\delta - W_\delta)\|^a_{\mathrm{L}^a((0,\delta);\mathrm{H}^m)} + \|(R_\delta - \Theta_\delta)\|^a_{\mathrm{L}^a((0,\delta);\mathrm{H}^{m+1})} = \mathscr{O}(\delta) \tag{3.26}$$

as $\delta \longrightarrow 0$; *e.g.*, for the second term this can be seen via

$$\begin{aligned}\|(R_\delta - \Theta_\delta)\|^a_{\mathrm{L}^a((0,\delta);\mathrm{H}^{m+1})}&\\ &= \int_0^\delta \|\Theta_0 - \delta^{-1}s\Delta\widetilde{\xi} + \delta^{-1}s\left(\boldsymbol{\Upsilon}\left(W_0 - \frac{\delta^{-1}s\partial_1\widetilde{\xi}}{2}, \boldsymbol{0}\right)\cdot\boldsymbol{\nabla}\right)\widetilde{\xi}\|^a_{m+1}\,\mathrm{d}s\\ &\leq \delta C\left(1 + \|W_0\|^{2a}_m + \|\Theta_0\|^a_{m+1} + \|\widetilde{\xi}\|^{2a}_{m+4}\right).\end{aligned}$$

The argument is then completed by utilizing (3.23), (3.25), and (3.26) to derive energy estimates for R_δ and Q_δ. Indeed, one can begin with formally multiplying (3.23) and (3.25) by $(-\Delta)^{m-1}Q_\delta$ and $(-\Delta)^m R_\delta$ respectively; integration by parts, Sobolev embeddings, and Grönwall's inequality subsequently imply, together with (3.26), that $\|Q_\delta(\cdot,\delta)\|_{m-1} \longrightarrow 0$ as $\delta \longrightarrow 0$. From the viewpoint of estimates, the procedure is similar to that presented in "Step 2" of the proof for Theorem 3.2.

Showing (3.20)**.** Instead of (3.21) we consider the time evolution of the modified trajectory

$$(W_\delta, \Theta_\delta) \coloneqq S_\delta(w_0 + \delta^{-1/2}\widehat{\xi}, \theta_0, \varphi_{\mathrm{ext}}, \psi_{\mathrm{ext}}, \boldsymbol{0}) - (\delta^{-1/2}\widehat{\xi}, 0),$$

which solves the problem

$$
\begin{aligned}
\partial_t W_\delta - \nu\Delta W_\delta + (\boldsymbol{U}_\delta \cdot \boldsymbol{\nabla})\, W_\delta = \varphi_{\text{ext}} + \partial_1 \Theta_\delta + \nu\delta^{-1/2}\Delta\widehat{\xi}\\
-\delta^{-1/2}\,(\boldsymbol{U}_\delta \cdot \boldsymbol{\nabla})\,\widehat{\xi} - \delta^{-1/2}\left(\boldsymbol{\Upsilon}(\widehat{\xi},\boldsymbol{0}) \cdot \boldsymbol{\nabla}\right) W_\delta - \delta^{-1}\left(\boldsymbol{\Upsilon}(\widehat{\xi},\boldsymbol{0}) \cdot \boldsymbol{\nabla}\right)\widehat{\xi},\\
\boldsymbol{\nabla} \wedge \boldsymbol{U}_\delta = W_\delta, \quad \boldsymbol{\nabla} \cdot \boldsymbol{U}_\delta = 0, \quad \int_{\mathbb{T}^2} \boldsymbol{U}_\delta(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x} = \boldsymbol{0},\\
\partial_t \Theta_\delta - \tau\Delta\Theta_\delta + (\boldsymbol{U}_\delta \cdot \boldsymbol{\nabla})\Theta_\delta = \psi_{\text{ext}} - \delta^{-1/2}\left(\boldsymbol{\Upsilon}(\widehat{\xi},\boldsymbol{0}) \cdot \boldsymbol{\nabla}\right)\Theta_\delta,\\
W_\delta(\cdot,0) = w_0, \quad \Theta_\delta(\cdot,0) = \theta_0.
\end{aligned}
$$

Then, the convergence in (3.20) follows after showing that the remainder terms

$$
\begin{aligned}
Q_\delta(\cdot,t) \coloneqq W_\delta(\cdot,t) - W_0 + \delta^{-1}t\left(\boldsymbol{\Upsilon}(\widehat{\xi},\boldsymbol{0}) \cdot \boldsymbol{\nabla}\right)\widehat{\xi} - \delta^{-1/2}t\nu\Delta\widehat{\xi},\\
R_\delta(\cdot,t) \coloneqq \Theta_\delta(\cdot,t) - \Theta_0
\end{aligned}
$$

satisfy $(Q_\delta, R_\delta)(\cdot,\delta) \longrightarrow (0,0)$ in $\mathrm{H}^{m-1} \times \mathrm{H}^m$ as $\delta \longrightarrow 0$. Indeed, instead of (3.26), one now has

$$
\|(Q_\delta - W_\delta)\|^a_{\mathrm{L}^a((0,\delta);\mathrm{H}^m)} + \|(R_\delta - \Theta_\delta)\|^a_{\mathrm{L}^a((0,\delta);\mathrm{H}^{m+1})} = \mathcal{O}(\delta^{1/2}),
$$

which suffices in order to derive standard energy estimates for Q_δ and R_δ as explained in the proof of Theorem 3.2. □

We also need the below auxiliary result which can be proved by elementary trigonometric calculations.

Lemma 3.5. *Let $\mathcal{E}$ be the collection of $\sin(\boldsymbol{x}\cdot\boldsymbol{n})$ and $\cos(\boldsymbol{x}\cdot\boldsymbol{n})$ with $\boldsymbol{n} \in \mathbb{N}\times(\mathbb{N}\cup\{0\})$. Then, the set*

$$
\mathcal{H} \coloneqq \left\{\xi_0 + (\boldsymbol{\Upsilon}(\xi_1,\boldsymbol{0}) \cdot \boldsymbol{\nabla})\,\xi_1 + (\boldsymbol{\Upsilon}(\xi_2,\boldsymbol{0}) \cdot \boldsymbol{\nabla})\,\xi_2 \mid \xi_0,\xi_1,\xi_2 \in \operatorname{span}_{\mathbb{R}} \mathcal{E}\right\}
$$

contains $\pm\sin(\boldsymbol{x}\cdot\boldsymbol{n})$ and $\pm\cos(\boldsymbol{x}\cdot\boldsymbol{n})$ for all nonzero $\boldsymbol{n} \in \mathbb{Z}^2$.

Proof. The idea is to utilize the representations $\boldsymbol{\Upsilon}(\sin(\boldsymbol{n}\cdot\boldsymbol{x}),\boldsymbol{0}) = \boldsymbol{n}^\perp|\boldsymbol{n}|^{-2}\cos(\boldsymbol{n}\cdot\boldsymbol{x})$ and $\boldsymbol{\Upsilon}(\cos(\boldsymbol{n}\cdot\boldsymbol{x}),\boldsymbol{0}) = -\boldsymbol{n}^\perp|\boldsymbol{n}|^{-2}\sin(\boldsymbol{n}\cdot\boldsymbol{x})$, and then to express $\sin(\boldsymbol{x}\cdot\boldsymbol{n})$ and $\cos(\boldsymbol{x}\cdot\boldsymbol{n})$ via trigonometric angle identities as elements of $\mathcal{H}$. Such arguments are described, *e.g.*, in [3]. □

As a direct consequence of Theorem 3.3 and Theorem 3.4, the following corollary allows to approximately reach all target states that are of the form “initial state + bilinear term + large residual with vanishing x_1-average”.

Corollary 3.6. *Given $m \geq 2$, $T > 0$, $\xi = \partial_1\kappa$ for some $\kappa \in \mathrm{C}^\infty(\mathbb{T}^2;\mathbb{R})$, initial states $(w_0,\theta_0) \in \mathrm{H}^m \times \mathrm{H}^{m+1}$, and $(\varphi_{\text{ext}},\psi_{\text{ext}}) \in \mathrm{L}^2((0,T);\mathrm{H}^{m-2}\times\mathrm{H}^{m-1})$. There exist control parameters $(\widetilde{\gamma}_\delta,\widetilde{\gamma}_{\delta,1},\ldots,\widetilde{\gamma}_{\delta,6})_{\delta>0} \subset \mathrm{L}^2((0,T);\mathbb{R})$ and $\widetilde{\boldsymbol{\aleph}}_\delta \in \mathrm{C}^\infty_0((0,T);\mathbb{R})$ such that*

$$
\Pi_1 S_\delta\left(w_0,\theta_0,\varphi_{\text{ext}},\psi_{\text{ext}} + \widetilde{\eta}_\delta,\widetilde{\boldsymbol{\aleph}}_\delta\right)|_{t=\delta} - \delta^{-1/2}\xi \longrightarrow w_0 - (\boldsymbol{\Upsilon}(\xi,\boldsymbol{0}) \cdot \boldsymbol{\nabla})\xi \text{ as } \delta \longrightarrow 0, \tag{3.27}
$$

where $\widetilde{\eta}_\delta(\boldsymbol{x},t) \coloneqq \sum_{l=1}^6 \widetilde{\gamma}_{\delta,l}(t)\zeta_l(\boldsymbol{x},\widetilde{\gamma}_\delta(t))$.

Proof. The argument consists of three steps. 1) For any $\delta_2 > 0$ and $\delta_3 > 0$, one can determine $\delta_1 > 0$ via Theorem 3.3 such that the corresponding trajectory in Theorem 3.3 comes at $t = \delta_1$ as close to $\theta_0 - \delta_2^{-1}\delta_3^{-1/2}\kappa$ as desired. 2) depending on δ_3, Theorem 3.4 allows taking δ_2 so small that the left-hand side in (3.19) with $\delta = \delta_2$ and $\widetilde{\xi} = \delta_3^{-1/2}\kappa$ is at $t = \delta_2$ as close to $w_0 - \delta_3^{-1/2}\xi$ as needed. 3) A good value of δ_3 is then taken so that the left-hand side in (3.20) with $\delta = \delta_3$ and $\widehat{\xi} = \xi$ is desirably near to $w_0 - (\boldsymbol{\Upsilon}(\xi, \mathbf{0}) \cdot \boldsymbol{\nabla})\xi$ at time $t = \delta_3$. Using the continuous dependence of solutions to (3.2) on the data, one can glue the respective trajectories from these steps by first fixing δ_3, then δ_2, and finally δ_1.

Summarized, via Theorem 3.3 and Theorem 3.4, we choose for any given $\delta > 0$ the numbers $0 < \delta_1, \delta_2, \delta_3 < \delta/3$, the parameters $(\alpha_{\delta_1,l})_{l\in\{1,\dots,6\}} \subset \mathrm{L}^2((0,1);\mathbb{R})$, and

$$\widetilde{\gamma}_\delta(t) \coloneqq \mathbb{I}_{(0,\delta_1)}\delta_1^{-1}t, \quad \widetilde{\gamma}_{\delta,l}(t) \coloneqq \delta_1^{-2}\alpha_{\delta_1,l}(\delta_1^{-1}t), \quad \widetilde{\boldsymbol{\aleph}}_\delta(t) \coloneqq \mathbb{I}_{(0,\delta_1)}\overline{\boldsymbol{y}}_{\delta_1}(t)$$

such that (3.27) holds. ◻

The next theorem provides the combined global approximate controllability of the vorticity and the temperature in any given time.

Theorem 3.7. *Assume that $m \geq 2$, $T > 0$, $\varepsilon > 0$, $(w_0, \theta_0), (w_T, \theta_T) \in \mathrm{H}^{m-1} \times \mathrm{H}^m$, and $(\varphi_{\mathrm{ext}}, \psi_{\mathrm{ext}}) \in \mathrm{L}^2((0,T); \mathrm{H}^{m-1} \times \mathrm{H}^m)$. There exist $\gamma, \gamma_1, \dots, \gamma_6 \in \mathrm{L}^2((0,T);\mathbb{R})$ such that the corresponding solution $(w, \boldsymbol{u}, \theta)$ to (3.2), with control η as in (1.5), has the regularity $(w, \theta) \in \mathcal{X}_T^m$ with $\fint_{\mathbb{T}^2} \boldsymbol{u}(\boldsymbol{x}, \cdot)\,\mathrm{d}\boldsymbol{x} \in \mathrm{C}_0^\infty((0,T);\mathbb{R}^2)$ and obeys*

$$\|w(\cdot,T) - w_T\|_{m-1} + \|\theta(\cdot,T) - \theta_T\|_m < \frac{\varepsilon}{C_0} \tag{3.28}$$

for the constant $C_0 > 0$ from (3.4).

Proof. By standard energy estimates for the vorticity-temperature system (3.2), there exists a small time $\widetilde{\delta_0} > 0$ such that the implication

$$\|a - w_T\|_{m-1} + \|b - \theta_T\|_m < \frac{2\varepsilon}{3C_0} \quad \Longrightarrow \quad \|w^\delta - w_T\|_{m-1} + \|\theta^\delta - \theta_T\|_m < \frac{\varepsilon}{C_0}$$

is true for all pairs

$$(w^\delta, \theta^\delta) \coloneqq S_\delta(a, b, \varphi_{\mathrm{ext}}(T - \delta + \cdot), \psi_{\mathrm{ext}}(T - \delta + \cdot), \mathbf{0})|_{t=\delta}, \quad 0 < \delta \leq \widetilde{\delta_0},$$

and where $\widetilde{\delta_0}$ depends on ε, C_0, w_T, θ_T, and $(\varphi_{\mathrm{ext}}, \psi_{\mathrm{ext}})$. Now, the desired parameters $\gamma, \overline{\gamma}, \gamma_1, \dots, \gamma_6 \in \mathrm{L}^2((0,T);\mathbb{R})$ are obtained as follows.

Step 1. Starting with controls switched off. During the time interval $[0, T - \widetilde{\delta_0}]$, no controls shall be applied. Let us denote the state of the uncontrolled trajectory at the time $t = T - \delta_0$ by

$$(\widetilde{w}_0, \widetilde{\theta}_0) \coloneqq S_{T-\delta_0}(w_0, \theta_0, \varphi_{\mathrm{ext}}, \psi_{\mathrm{ext}}, \mathbf{0})|_{t=T-\delta_0} \in \mathrm{H}^{m+1} \times \mathrm{H}^{m+2},$$

where $0 < \delta_0 \leq \widetilde{\delta}_0$ is arbitrarily selected outside a temporal set of zero measure on which the desired regularity $(\widetilde{w}_0, \widetilde{\theta}_0) \in \mathrm{H}^{m+1} \times \mathrm{H}^{m+2}$ is unknown. This is possible due to the assumption $(\varphi_{\mathrm{ext}}, \psi_{\mathrm{ext}}) \in \mathrm{L}^2((0,T);\mathrm{H}^{m-1} \times \mathrm{H}^m)$, as it follows then from the parabolic smoothing effects of (3.2) (*cf.* [22]) that

$$S_T(w_0, \theta_0, \varphi_{\mathrm{ext}}, \psi_{\mathrm{ext}}, \mathbf{0})|_{(a,T)} \in \mathrm{L}^2((a,T);\mathrm{H}^{m+1} \times \mathrm{H}^{m+2})$$

for arbitrarily fixed $0 < a < T$.

Step 2. Energizing special intermediate profiles. Owing to Lemma 3.5, we choose an integer $L \geq 0$ and average-free $\xi_0, \xi_1, \dots, \xi_{2L} \in \mathrm{C}^\infty(\mathbb{T}^2;\mathbb{R})$, where $\xi_i = \partial_1 \kappa_i$ for some $\kappa_i \in \mathrm{C}^\infty(\mathbb{T}^2;\mathbb{R})$ and $i \in \{1, \dots, 2L\}$, so that

$$\|V - w_T\|_{m-1} < \frac{\varepsilon}{6C_0}, \quad V \coloneqq \widetilde{w}_0 - \partial_1 \xi_0 - \sum_{i=1}^{2L} (\boldsymbol{\Upsilon}(\xi_i, \mathbf{0}) \cdot \boldsymbol{\nabla})\xi_i. \tag{3.29}$$

Now, by $2L$ applications of Corollary 3.6, starting from $(\widetilde{w}_0, \widetilde{\theta}_0)$ one can approximately reach the state comprised of $\sum_{i=1}^{2L} (\boldsymbol{\Upsilon}(\xi_i, \mathbf{0}) \cdot \boldsymbol{\nabla})\xi_i$ plus $2L$ large residuals which arise from the "$\delta^{-1/2}\xi = \delta^{-1/2}\partial_1\kappa$" term in (3.27). Then, resorting once more to Theorem 3.3 and the statement (3.19) of Theorem 3.4, the $-\partial_1\xi_0$ term in (3.29) can be generated while eliminating the aforementioned residuals. Thus, one obtains $\delta_1 < \delta_0/2$ and parameters $(\widetilde{\gamma}, \widetilde{\gamma}_1, \dots, \widetilde{\gamma}_6)_{\delta > 0} \in \mathrm{L}^2((0,T);\mathbb{R})$ and $\widetilde{\boldsymbol{\aleph}} \in \mathrm{C}_0^\infty((0,T);\mathbb{R})$ such that

$$(\widetilde{w}, \widetilde{\theta}) \coloneqq S\left(\widetilde{w}_0, \widetilde{\theta}_0, \varphi_{\mathrm{ext}}, \psi_{\mathrm{ext}} + \widetilde{\eta}, \widetilde{\boldsymbol{\aleph}_\delta}\right)|_{t=\delta_1},$$

where

$$\widetilde{\eta}(\boldsymbol{x}, t) \coloneqq \sum_{l=1}^{6} \widetilde{\gamma}_l(t)\zeta_l(\boldsymbol{x}, \widetilde{\gamma}(t)),$$

obeys

$$(\widehat{w}_0, \widehat{\theta}_0) \coloneqq (\widetilde{w}, \widetilde{\theta})(\cdot, \delta_1) \in \mathrm{H}^m \times \mathrm{H}^{m+1}, \quad \|\widetilde{w}(\cdot, \delta_1) - V\|_{m-1} < \frac{\varepsilon}{6C_0}. \tag{3.30}$$

The regularity condition in (3.30) holds due to $(\varphi_{\mathrm{ext}}, \psi_{\mathrm{ext}}) \in \mathrm{L}^2((0,T);\mathrm{H}^{m-1} \times \mathrm{H}^m)$ and parabolic smoothing effects.

Step 3. Reaching the target state. Another application of Theorem 3.3, with any target temperature $\widehat{\theta}_T \in \mathrm{H}^{m+1}$ satisfying $\|\widehat{\theta}_T - \theta_T\|_m < \varepsilon/6C_0$, provides $0 < \delta_2 < \delta_0/2$ and control parameters

$$(\beta_1, \dots, \beta_6) \subset \mathrm{L}^2((0,1);\mathbb{R})$$

for which the trajectory

$$(\widehat{w}, \widehat{\theta}) \coloneqq S_{\delta_2}(\widehat{w}_0, \widehat{\theta}_0, \varphi_{\mathrm{ext}}(T - \delta_0 + \delta_1 + \cdot), \psi_{\mathrm{ext}}(T - \delta_0 + \delta_1 + \cdot) + \widehat{\eta}, \overline{\boldsymbol{y}}_{\delta_2})$$

driven by $\widehat{\eta}(\cdot,t) := \delta_2^{-1}\sum_{l=1}^{6}\beta_l(\delta_2^{-1}t)\zeta_l(\cdot,\delta_2^{-1}t)$ for $0 \leq t \leq \delta_2$ satisfies

$$\|\widehat{w}(\cdot,\delta_2) - \widehat{w}_0\|_{m-1} + \|\widehat{\theta}(\cdot,\delta_2) - \widehat{\theta}_T\|_m < \frac{\varepsilon}{3C_0}.$$

In view of (3.29) and (3.30), it follows that

$$\|\widehat{w}(\cdot,\delta_2) - w_T\|_{m-1} + \|\widehat{\theta}(\cdot,\delta_2) - \theta_T\|_m < \frac{2\varepsilon}{3C_0}.$$

Summary. By gluing together the above-obtained trajectories and controls, the previous constructions yield the control parameters $\gamma, \gamma_1, \ldots, \gamma_6 \in \mathrm{L}^2((0,T);\mathbb{R})$ and a respectively controlled trajectory $(w, \boldsymbol{u}, \theta)$ having the desired properties. In particular, since $\overline{\boldsymbol{y}}$ obtained via Theorem 2.1 is compactly supported in $(0,1)$, it follows that $\fint_{\mathbb{T}^2} \boldsymbol{u}(\boldsymbol{x},\cdot)\,\mathrm{d}\boldsymbol{x} \in \mathrm{C}_0^\infty((0,T);\mathbb{R}^2)$; and in fact, the first component of the velocity average vanishes: $\fint_{\mathbb{T}^2} u_1(\boldsymbol{x},\cdot)\,\mathrm{d}\boldsymbol{x} = 0$. □

3.3 Conclusion of Theorems 1.2 and 1.3

Let $r \geq 2$, $T > 0$, and $\varepsilon > 0$ be as in Theorems 1.2 and 1.3. An application of Theorem 3.7 with $m = r$ and $(w_0, w_T) := (\boldsymbol{\nabla} \wedge \boldsymbol{u}_0, \boldsymbol{\nabla} \wedge \boldsymbol{u}_T)$ provides a solution $(\boldsymbol{u}, w, \theta)$ to (3.2) that satisfies (3.28), and which is driven by a control η of the form

$$\eta(\boldsymbol{x},t) = \sum_{l=1}^{6} \gamma_l(t)\zeta_l(\boldsymbol{x},\gamma(t)).$$

An issue is now that, by the proof of Theorem 3.7, $\boldsymbol{u}(\cdot,t)$ should have nonzero (large) average for some $t \in (0,T)$, but θ obtained in Theorem 3.7 as a combination of Theorem 3.3 and Theorem 3.4 is average-free (see (3.13)). Thus, in order to pass to the velocity-temperature formulation, we shall now either add a velocity control, or modify θ appropriately. Hereto, let us denote by $\aleph \in \mathrm{C}_0^\infty((0,T);\mathbb{R})$ the function determined by $\aleph \boldsymbol{e}_{\mathrm{grav}} = \fint_{\mathbb{T}^2} \boldsymbol{u}(\boldsymbol{x},\cdot)\,\mathrm{d}\boldsymbol{x} \in \mathrm{C}_0^\infty((0,T);\mathbb{R}^2)$.

Option 1. Using only a localized temperature control. Let χ be the cutoff from (2.1) with $\operatorname{supp}(\boldsymbol{x} \mapsto \chi(x_2)) \subset \omega$. Then, we replace θ by $\widetilde{\theta}(\boldsymbol{x},t) := \theta(\boldsymbol{x},t) + \chi(x_2)\aleph'(t)$, which satisfies (*cf.* Theorem 3.7)

$$\widetilde{\theta}(\cdot,0) = \theta(\cdot,0), \quad \widetilde{\theta}(\cdot,T) = \theta(\cdot,T),$$

followed by renaming $\widetilde{\theta}$ again as θ. Then, the triple $(\boldsymbol{u} = [u_1, u_2]^\top, w, \theta)$ obeys (3.28) and solves (1.1) with a control η of the form

$$\eta(\boldsymbol{x}) = \chi(x_2)\aleph''(t) - \tau\chi''(x_2)\aleph'(t) + u_2(\boldsymbol{x},t)\chi'(x_2)\aleph'(t) + \sum_{l=1}^{6} \gamma_l(t)\zeta_l(\boldsymbol{x},\gamma(t)),$$

which satisfies $\operatorname{supp}(\eta) \subset \omega$. As seen in the proof of Theorem 2.7, we can take $\zeta_1 = \chi$ and $\zeta_2 = \chi'$. This leads to the resolution of Theorem 1.2.

Remark 3.8. One could also correct the temperature average by adding $\aleph'(t)$ to θ obtained from Theorem 3.7. While this would allow to merely employ a finitely decomposable temperature control, there appears one term that is not physically localized in ω: $\mathbb{I}_\omega \eta(\boldsymbol{x},t)$ in (1.1) would be replaced by $\aleph''(t) + \mathbb{I}_\omega \sum_{l=1}^{6} \gamma_l(t)\zeta_l(\boldsymbol{x},\gamma(t))$.

Option 2. One-dimensional control in the second velocity component. The triple $(\boldsymbol{u}, w, \theta)$ obtained via Theorem 3.7 satisfies the controllability condition (3.28) and solves

$$\partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} + \boldsymbol{\nabla} p = (\theta + \mathbb{I}_\omega\overline{\eta})\boldsymbol{e}_{\mathrm{grav}} + \boldsymbol{\Phi}_{\mathrm{ext}}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \quad w(\cdot,0) = w_0,$$

$$\partial_t \theta - \tau\Delta\theta + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\theta = \mathbb{I}_\omega\left(\sum_{l=1}^{6}\gamma_l(t)\zeta_l(\boldsymbol{x},\gamma(t))\right) + \psi_{\mathrm{ext}}, \quad \theta(\cdot,0) = \theta_0,$$

where $\overline{\eta}(\boldsymbol{x},t) \coloneqq \aleph'(t)\chi(x_2)$, and noting that χ is a universal profile which only depends on ω. Because in (2.20) the there-appearing cutoff profile χ can be taken identically to the one that is introduced here with the same name, we may write $\overline{\eta}(\boldsymbol{x},t) = \aleph'(t)\zeta_1(x_2)$. This completes the proof of Theorem 1.3.

Conclusion. Since $\boldsymbol{x} \mapsto \chi(x_2)\boldsymbol{e}_{\mathrm{grav}}$ is curl-free, all above-listed options for the controls ensure together with the relations in (3.4) and (3.28) the target condition

$$\|\boldsymbol{u}(\cdot,T) - \boldsymbol{u}_T\|_r + \|\theta(\cdot,T) - \theta_T\|_r \leq C_0\|w(\cdot,T) - w_T\|_{r-1} + \|\theta(\cdot,T) - \theta_T\|_r < \varepsilon.$$

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