Controllability of Boussinesq flows driven by finite-dimensional and physically localized forces

Manuel Rissel*

Abstract

We show approximate controllability of Boussinesq flows in $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ driven by finite-dimensional controls that are supported in any fixed region $\omega \subset \mathbb{T}^2$. This addresses a Boussinesq version of a question by Agrachev and provides a first example of fluid PDEs controllable in that sense. In particular, we add in this context to results obtained for the Navier–Stokes system by Agrachev-Sarychev (Comm. Math. Phys. 265, 2006), where the controls are finite-dimensional but not localized in physical space, and Nersesyan-Rissel (Comm. Pure Appl. Math. 78, 2025), where physically localized controls admit for special ω a degenerate but not finite-dimensional structure.

For our proof, we study controllability properties of tailored convection equations governed by time-periodic degenerately forced Euler flows that provide a twofold geometric mechanism: transport of information through ω versus non-stationary mixing effects transferring energy from low-dimensional sources to higher frequencies. The temperature is then controlled by using Coron's return method, while the velocity is mainly driven by the buoyant force.

When ω contains two cuts of the torus and a closed square of side-length L, our approach yields explicit control spaces for the velocity and temperature of dimensions $2 + 18\lceil 2\pi/L\rceil^2 + 8\lceil 2\pi/L\rceil^4$ and $2 + 8\lceil 2\pi/L\rceil^2$, respectively.

Keywords

Boussinesq system, incompressible fluids, approximate controllability, finite-dimensional controls, physically localized controls

MSC2020

35Q30, 35Q35, 76B75, 80A19, 93B05, 93C10

*NYU-ECNU Institute of Mathematical Sciences at NYU Shanghai, 3663 Zhongshan Road North, Shanghai, 200062, China, e-mail: Manuel.Rissel@nyu.edu

Contents

1 Introduction

Let $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ and fix a nonempty open set $\omega \subset \mathbb{T}^2$. The objective of this work is to study the propagation of degenerate forces localized in ω under the dynamics of Boussinesq flows to rich sets in the state space. Given any time $T > 0$, we consider the velocity u, temperature θ, and pressure p describing the motion of an incompressible fluid modeled by the 2D Boussinesq system. That is, $u\colon \mathbb{T}^2 \times (0,T) \longrightarrow \mathbb{R}^2$ and $\theta, p\colon \mathbb{T}^2 \times (0,T) \longrightarrow \mathbb{R}$ satisfy in $\mathbb{T}^2 \times (0,T)$ the initial value problem

$$\begin{aligned} \partial_t u - \nu\Delta u + (u\cdot\nabla)\, u + \nabla p &= \theta e_2 + f + \mathbb{I}_\omega \xi, \\ \operatorname{div}(u) &= 0, \\ \partial_t\theta - \tau\Delta\theta + (u\cdot\nabla)\theta &= g + \mathbb{I}_\omega \eta, \\ u(\cdot,0) = u_0, &\quad \theta(\cdot,0) = \theta_0, \end{aligned} \tag{1.1}$$

where $\nu > 0$ and $\tau > 0$ specify the viscosity and thermal diffusivity, $\mathbb{I}_S$ denotes the indicator function of a set S, the unit vector $e_2 = (0, 1)$ points in the direction of gravity, u_0 and θ_0 are the initial states, f and g are known forces, and ξ and η are the to-be-determined controls. For further background on the Boussinesq system, which is relevant to the study of geophysical phenomena, turbulent flows, and other topics, we refer to [7, 21, 35].

As made precise in Corollary 1.2 below, the meaning of approximate controllability of (1.1) can be sketched as follows. For any approximation accuracy $\varepsilon > 0$, time $T > 0$, initial- and target states (u_0, θ_0) and (u_1, θ_1), parameters $\nu, \tau > 0$, and forces (f, g), there exist controls (ξ, η) such that

$$\|u(\cdot, T) - u_1\| + \|\theta(\cdot, T) - \theta_1\| < \varepsilon,$$

where $\|\cdot\|$ denotes suitable norms. In particular, this is a global (large data) notion of controllability.

To achieve this with finite-dimensional controls means that there are universal numbers $d_1, d_2 \in \mathbb{N}$ and functions $\xi_1, \dots, \xi_{d_1} \colon \mathbb{T}^2 \longrightarrow \mathbb{R}^2$ and $\eta_1, \dots, \eta_{d_2} \colon \mathbb{T}^2 \longrightarrow \mathbb{R}$, which depend on the fixed choice of ω but are independent of ν, τ, ε, and all data in (1.1), such that ξ and η have for each $t \in [0, T]$ the form

$$\begin{aligned} \xi(\cdot, t) &= \alpha_1(t)\xi_1 + \dots + \alpha_{d_1}(t)\xi_{d_1}, \\ \eta(\cdot, t) &= \beta_1(t)\eta_1 + \dots + \beta_{d_2}(t)\eta_{d_2} \end{aligned}$$

with control coefficients

$$\alpha_1, \dots \alpha_{d_1}, \beta_1, \dots, \beta_{d_2} \colon [0, T] \longrightarrow \mathbb{R}.$$

The numbers d_1, d_2 and profiles $\xi_1, \dots, \xi_{d_1}, \eta_1, \dots, \eta_{d_2}$ must remain unchanged when varying the viscosity, thermal diffusivity, approximation accuracy, initial- and target states, and prescribed body forces. Only the control coefficients $\alpha_1, \dots \alpha_{d_1}$ and $\beta_1, \dots, \beta_{d_2}$ in the representations of ξ and η can be chosen in dependence on the data in order to influence the final state of the solution to (1.1). This translates to the goal of constructing universal finite-dimensional vector spaces $\mathscr{F}_v \subset C^\infty(\mathbb{T}^2; \mathbb{R}^2)$ and $\mathscr{F}_t \subset C^\infty(\mathbb{T}^2; \mathbb{R})$ of functions supported in ω such that (1.1) is approximately controllable in time $T > 0$ with controls that satisfy $\xi(\cdot, t) \in \mathscr{F}_v$ and $\eta(\cdot, t) \in \mathscr{F}_t$ for $t \in [0, T]$.

For the Navier–Stokes system on the torus driven by finite-dimensional but not physically localized forces, approximate controllability has been shown first in [2, 3] via the Agrachev-Sarychev method. Controllability by physically localized but not finite-dimensional controls on the torus is known due to [11, 24], where

Coron's return method is used. Whether approximate controllability holds with controls that are both finite-dimensional and physically localized constitutes an open problem posed by Agrachev for the Navier–Stokes system (*c.f.* [1, Section 7]). Also for other fluid models, questions of this type have remained unanswered until the present work. Here, we give a positive answer for the planar Boussinesq system.

1.1 Notation

Function spaces. Given $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and writing H_{avg} for the $L^2(\mathbb{T}^2;\mathbb{R})$-functions with zero average, we denote the L^2-based Sobolev spaces of divergence-free vector fields and of zero average scalar functions

$$H := \left\{ f \in H^2_{\mathrm{avg}} \mid \nabla \cdot f = 0 \text{ in } \mathbb{T}^2 \right\}, \quad V^m := H^m(\mathbb{T}^2;\mathbb{R}^2) \cap H,$$
$$H^m := H^m(\mathbb{T}^2;\mathbb{R}) \cap H_{\mathrm{avg}},$$

where H^m and V^m are endowed with the usual norms $\|\cdot\|_m$ of $H^m(\mathbb{T}^2;\mathbb{R})$ and $H^m(\mathbb{T}^2;\mathbb{R}^2)$ respectively. Further, we say that $f \in L^2((0,T);C^\infty(\mathbb{T}^2;\mathbb{R}^N))$, with $T > 0$ and $N \in \{1,2\}$, when $f \in L^2((0,T);H^m(\mathbb{T}^2;\mathbb{R}^N))$ for all $m \in \mathbb{N}$. Throughout, the Lebesgue measure is normalized such that $\int_{\mathbb{T}^2} dx = 1$.

Flow maps. Let $T > 0$ and v be a continuous map $\mathbb{T}^2 \times [0,T] \longrightarrow \mathbb{R}^2$ that is Lipschitz continuous in the space variables with time-independent Lipschitz constant. Then, the Cauchy–Lipschitz theorem provides for each $x \in \mathbb{T}^2$ and $s \in [0,T]$ a unique solution $\Phi^v(x,s,\cdot)\colon [0,T] \longrightarrow \mathbb{T}^2$ to the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi^v(x,s,t) = v(\Phi^v(x,s,t),t), \quad \Phi^v(x,s,s) = x. \tag{1.2}$$

We call Φ^v the flow of v and note that $\Phi^v(\Phi^v(x,s,r),r,t) = \Phi^v(x,s,t)$ for all $x \in \mathbb{T}^2$ and $r,s,t \in [0,T]$. When $A \subset \mathbb{T}^2$ and $I,J \subset [0,T]$, we write $\Phi^v(A,I,J) := \{\Phi^v(x,s,t) \,|\, x \in A,\ s \in I,\ t \in J\}$.

Div-curl problems. For sufficiently regular $U = (U_1,U_2)\colon \mathbb{T}^2 \longrightarrow \mathbb{R}^2$, the "curl" of U is defined as $\nabla \wedge U := \partial_1 U_2 - \partial_2 U_1$. Moreover, for $z \in H^m$, $m \in \mathbb{N}_0$, and $A \in \mathbb{R}^2$, we denote by $\Upsilon(z,A) \in H^{m+1}(\mathbb{T}^2;\mathbb{R}^2)$ the unique solution to the div-curl problem

$$\nabla \cdot \Upsilon(z,A) = 0, \quad \nabla \wedge \Upsilon(z,A) = z \tag{1.3}$$

that satisfies $\int_{\mathbb{T}^2} \Upsilon(z,A)(x)\,dx = A$. One can express Υ as $\Upsilon(z,A) = \nabla^\perp \phi + A$, where the stream function ϕ solves Poisson's equation $\Delta\phi = -z$ in $\mathbb{T}^2$ and $\nabla^\perp\phi := (\partial_2\phi, -\partial_1\phi)$. When $A = 0$, we abbreviate $\Upsilon(z) = \Upsilon(z,0)$.

1.2 Main results

The following Theorem is our main contribution. It provides quick approximate controllability for the temperature, while keeping the velocity close to its initial state. At this point, additional regularity properties of the data are assumed; these assumptions will be relaxed due the parabolic smoothing effects exhibited by (1.1). The full approximate controllability of the Boussinesq system is stated below in Corollary 1.2.

Theorem 1.1. *There are finite-dimensional spaces $\mathscr{F}_v \subset C^\infty(\mathbb{T}^2;\mathbb{R}^2)$ and $\mathscr{F}_t \subset C^\infty(\mathbb{T}^2;\mathbb{R}) \cap H_{\mathrm{avg}}$ such that the following statement holds. For any given data*

$$\nu, \tau, \varepsilon, T > 0, \quad m \in \mathbb{N}, \quad u_0 \in V^{m+2}, \quad \theta_0, \theta_1 \in H^{m+2},$$
$$f \in L^2((0,T);V^m), \quad g \in L^2((0,T);H^m),$$

there exists $\delta_0 > 0$ so that for each $\delta \in (0,\delta_0)$ there are $\xi \in L^2((0,\delta);\mathscr{F}_v)$ and $\eta \in L^2((0,\delta);\mathscr{F}_t)$ for which the associated solution

$$(u,\theta) \in C^0([0,\delta];H^m(\mathbb{T}^2;\mathbb{R}^2) \times H^m) \cap L^2((0,\delta);H^{m+1}(\mathbb{T}^2;\mathbb{R}^2) \times H^{m+1})$$

to the Boussinesq problem (1.1) satisfies

$$\|u(\cdot,\delta) - u_0\|_{m+1} + \|\theta(\cdot,\delta) - \theta_1\|_{m+1} < \varepsilon.$$

The proof of Theorem 1.1 is organized as follows. In Section 2, a controllability result for transport problems with generating drift is recalled. In Section 3, approximate controllability via finite-dimensional and physically localized forces is established for a specially constructed convection problem. The argument is completed in Section 4.1.

As a corollary of Theorem 1.1, we can conclude also the approximate controllability of both the temperature and the velocity in arbitrary time, and for less regular initial states. A sketch of this argument, which will be given in more detail in Section 4.2, goes as follows.

1) Let $m \in \mathbb{N}$, $f \in L^2((0,T);V^m)$, and $g \in L^2((0,T);H^m)$. By the well-posedness of the 2D Boussinesq system, one can choose $\sigma > 0$ so small that, if an uncontrolled solution (u,θ) to (1.1) is issued at $t = t_0$ from the $\varepsilon/2$-neighborhood of (u_1,θ_1) in $V^{m+1} \times H^{m+1}$, then $(u,\theta)(\cdot,t)$ remains in the ε-neighborhood of (u_1,θ_1) in $H^{m+1}(\mathbb{T}^2;\mathbb{R}^2) \times H^{m+1}(\mathbb{T}^2;\mathbb{R})$ for all $t \in [t_0, t_0+\sigma]$. Thus, to control the system in any given time $T > 0$ and with less regular initial states, one can first issue a trajectory of (1.1) in the Leray-Hopf weak sense from $(u_0,\theta_0) \in H \times H_{\mathrm{avg}}$

at time $t = 0$ with zero controls ($\xi = 0$, $\eta = 0$). By parabolic regularization effects, available due to the choice of forces $f \in L^2((0,T);V^m)$ and $g \in L^2((0,T);H^m)$, this trajectory will belong to $V^{m+2} \times H^{m+2}$ at almost all times $t > 0$. Then, starting from $t = T - \sigma$ one employs the actual control strategy; we refer also to the similar situations in [38,40,41].

2) To steer also the velocity approximately to any given target, and not only the temperature, a mechanism from [40] can be applied, relying on several scaling limits and the fact that the set $\mathscr{E}$, defined as

$$\begin{gathered}\mathscr{E} \coloneqq \left\{q_0 + (\Upsilon(q_1)\cdot\nabla)\, q_1 + (\Upsilon(q_2)\cdot\nabla)\, q_2 \mid q_0, q_1, q_2 \in \operatorname{span}_{\mathbb{R}} \mathscr{E}_0\right\}, \\ \mathscr{E}_0 \coloneqq \{\sin(x\cdot n),\, \cos(x\cdot n) \mid n \in \mathbb{N}\times\mathbb{N}_0\},\end{gathered} \tag{1.4}$$

contains $\pm\sin(x\cdot n)$ and $\pm\cos(x\cdot n)$ for all $n \in \mathbb{Z}^2\setminus\{0\}$; *c.f.* [3] and also [40, Lemma 3.5]. More precisely, given any $q \in C^\infty(\mathbb{T}^2;\mathbb{R})$ with zero average, it is shown in [40, Theorem 3.4] that

$$\nabla\wedge u_\delta(\cdot,\delta) \longrightarrow \nabla\wedge u_0 - \partial_1 q \;\text{ in }\; H^m \;\text{ as }\; \delta \longrightarrow 0,$$

where $(u_\delta, \theta_\delta)$ solves (1.1) with zero controls $(\xi,\eta) = (0,0)$, initial velocity $u_0 \in H^{m+2}$, and initial temperatures of the form $\theta_0 = -\delta^{-1}q$. In addition, the latter reference provides in $H^m \times H^{m+1}$ the convergence

$$(\nabla\wedge u_\delta, \theta_\delta)(\cdot,\delta) - (\delta^{-1/2}q, 0) \longrightarrow (\widetilde{w}_0 - (\Upsilon(q)\cdot\nabla)q, \theta_0) \;\text{ as }\; \delta \longrightarrow 0,$$

where $(u_\delta, \theta_\delta)$ is the solution to (1.1) with zero controls $(\xi,\eta) = (0,0)$, initial temperature $\theta_0 \in H^{m+2}$, and initial vorticity $\nabla\wedge u_0 = \widetilde{w}_0 + \delta^{-1/2}q$ for a given $\widetilde{w}_0 \in H^{m+1}$. Combining iterations of these two convergence results and Theorem 1.1, one can steer $\nabla\wedge u$ arbitrarily fast, and as close as desired in H^m, from any $\widetilde{w}_0 \in H^{m+1}$ to any finite sum of the form

$$\widetilde{w}_0 - q_0 - \sum_{i=1}^{2N}(\Upsilon(q_i)\cdot\nabla)q_i,$$

where $N \in \mathbb{N}$ and $q_0, q_1, \ldots, q_{2N} \in \operatorname{span}_{\mathbb{R}} \mathscr{E}_0$. Owing to the form of $\mathscr{E}$, this implies approximate controllability for the vorticity. As both convergences from [40, Theorem 3.4] are uniform with respect to (f, g) from bounded subsets of $L^2((0,T);V^m\times H^m)$, one can as in [40] define in a piece-wise (in time) manner a suitably controlled trajectory. Due to Theorem 1.1, the resulting controls are here finite-dimensional and physically localized.

Corollary 1.2. *Let the spaces* $\mathscr{F}_v \subset C^\infty(\mathbb{T}^2;\mathbb{R}^2)$ *and* $\mathscr{F}_t \subset C^\infty(\mathbb{T}^2;\mathbb{R}) \cap H_{\mathrm{avg}}$ *be obtained via Theorem 1.1. For any given* $\varepsilon, \nu, \tau, T > 0$, $k \in \mathbb{N}_0$, $u_0 \in H$, $u_1 \in V^k$, $\theta_0 \in H_{\mathrm{avg}}$, $\theta_1 \in H^k$, $f \in L^2((0,T);V^{\max\{k-1,1\}})$, *and* $g \in L^2((0,T);H^{\max\{k-1,1\}})$, *there exist controls* $\xi \in L^2((0,T);\mathscr{F}_v)$ *and* $\eta \in L^2((0,T);\mathscr{F}_t)$ *such that the solution*

$$u \in C^0((0,T];H^{\max\{k,2\}}(\mathbb{T}^2;\mathbb{R}^2)) \cap L^2((0,T);H^{\max\{k+1,3\}}(\mathbb{T}^2;\mathbb{R}^2)),$$
$$\theta \in C^0((0,T];H^{\max\{k,2\}}) \cap L^2((0,T);H^{\max\{k+1,3\}})$$

to the Boussinesq problem (1.1) *satisfies*

$$\|u(\cdot,T) - u_1\|_k + \|\theta(\cdot,T) - \theta_1\|_k < \varepsilon.$$

Remark 1.3. To simplify the presentation of our contributions, we prescribe zero average initial- and target states in Theorem 1.1 and Corollary 1.2. However, our approach also works for initial- and target states of non-zero average, possibly requiring the addition of a two-dimensional space to $\mathscr{F}_v$ and of a one-dimensional space to $\mathscr{F}_t$. This is explained in Remark 4.7 at the end of Section 4.2. It is worth noting that velocity controls cannot be divergence-free in general, as this function class may leave the velocity average invariant. *E.g.*, for simply-connected ω, one has

$$\begin{aligned}&\operatorname{div}(\xi) = 0 \,\wedge\, \operatorname{supp}(\xi) \subset \omega \implies \exists \text{ periodic } \phi\colon \xi = \nabla^\perp \phi \\ &\quad\implies \frac{d}{dt}\int_{\mathbb{T}^2} u \cdot e_1 \, dx = \int_{\mathbb{T}^2} (\nu\Delta u \cdot e_1 - (u\cdot\nabla)\, u \cdot e_1 - \partial_1 p + \partial_2\phi)\, dx = 0,\end{aligned}$$

obstructing approximate controllability when $\int_{\mathbb{T}^2} u_0(x)\cdot e_1\, dx \neq \int_{\mathbb{T}^2} u_1(x)\cdot e_1\, dx$ for $e_1 = (1,0)$. The controls obtained here are in any case not divergence-free, which is however in alignment with the existing literature on controllability properties of incompressible fluids driven by physically localized forces; see also the references in Section 1.5.

Remark 1.4. The controls in Theorem 1.1 and Corollary 1.2 can be chosen smooth in time, using a density argument and the stability of solutions to the Boussinesq system with respect to small perturbations of the forces.

1.3 Overview of the approach

We start with a so-called "generating" vector field $\overline{u}^\star$ that has small uniform norm (depending only on ω) and is constructed from an observable family as described in Definition 2.3. This notion of observability, first introduced in [29] for the study of randomly forced PDEs, induces a certain type of non-stationary mixing effect, propagating energy created by low-dimensional forces to higher frequencies. Choosing $\overline{u}^\star$ of small norm ensures that its flow cannot transport information in a fixed time over large distances, which will be crucial for the definition of our localized controls. Then, having a linearized and inviscid version of the temperature equation from (1.1) in mind, we consider on a small time interval $[0, T^\star]$ the transport problem

$$\partial_t v + (\overline{u}^\star \cdot \nabla) v = g^\star \tag{1.5}$$

for which approximate controllability by means of low-dimensional controls $g^\star$ without physical localization is known; since $g^\star$ can act everywhere in $\mathbb{T}^2$, the small uniform norm of $\overline{u}^\star$ is not an obstruction. Based on this, we construct a universal vector field $\overline{U}$, depending only on ω, such that approximate controllability also holds for

$$\partial_t V + (\overline{U} \cdot \nabla) V = \mathbb{I}_\omega G \tag{1.6}$$

with a finite-dimensional control G. Up to a few technical details, the force G will on the reference time interval $[0, 1]$ be given by

$$G(x, t) = \mu(x) \sum_{i=1}^{M} \mathbb{I}_{[t_a^i, t_b^i]}(t) g^\star(x - S_i, t - t_a^i), \tag{1.7}$$

where $g^\star$ is a low-dimensional control for (1.5), M is a number depending only on the geometry of the control region, μ is a particular cutoff supported in ω, $[t_a^i, t_b^i] \subset (0, 1)$ are disjoint time intervals of length $T^\star$ on which $\overline{U}$ will represent a physically localized version of $\overline{u}^\star(\cdot - S_i, \cdot - t_a^i)$, and $S_i \in \mathbb{R}^2$ are fixed shifts related to the convection mechanism that $\overline{U}$ provides for times outside $[t_a^i, t_b^i]$. See also Figure 1.

The idea is, by a careful construction of $\overline{U}$ involving some geometric considerations, to achieve the rearrangement (see Theorem 3.13)

$$\int_0^{T^\star} g^\star(\Phi^{\overline{u}^\star}(x, T^\star, s), s)\, ds = \int_0^1 G(\Phi^{\overline{U}}(x, 1, s), s)\, ds,$$

where $\Phi^{\overline{u}^\star}$ and $\Phi^{\overline{U}}$ are the flows of the vector fields $\overline{u}^\star$ and $\overline{U}$, respectively. Next, through a hydrodynamic scaling limit, which is commonly part of the return method in the context of incompressible fluids (*c.f.* [10, Chapter 6]), our result for (1.6) yields finite-dimensional and physically localized controls that steer the temperature approximately to any target, while the velocity is kept near its initial state. To this

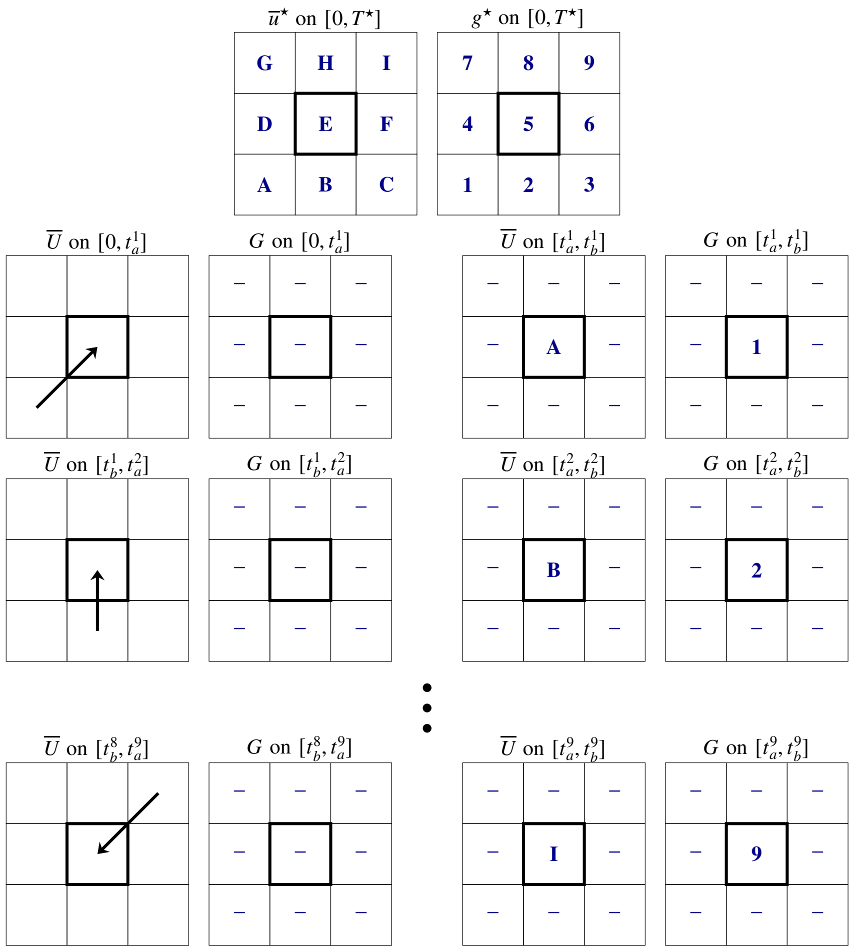

Figure 1: The behavioral patterns of $\overline{u}^\star$ and $g^\star$ from the convection problem (1.5) are schematically denoted on a subdivision of the torus by letters from A to I and numbers from 1 to 9, respectively. It is then indicated how G and $\overline{U}$ from (1.6) are obtained with the help of $g^\star$ and $\overline{u}^\star$. The arrows indicate how information is propagated along $\overline{U}$ into the control zone (bold center square). In regions marked with "–", the flows or controls are inactive.

end, the profile $\overline{U}$ will be used as a reference trajectory (in the return method sense), satisfying an incompressible Euler system driven by a finite-dimensional and physically localized force. In addition, $\overline{U}$ has to encode certain non-stationary mixing effects, like those provided by $\overline{u}^\star$, to guarantee approximate controllability of (1.6) by finite-dimensional forces. But, to localize the controls, $\overline{U}$ should also behave like a gradient flow in the complement of ω, transporting information into the control region.

Compared to the Navier–Stokes equations, we can exploit here the buoyant force to steer the velocity indirectly. To control the velocity directly by using the return method, we would need to obtain finite-dimensional and physically localized controls G not for (1.6), but instead for the following convection problem with nonlocal stretching term:

$$\partial_t V + (\overline{U} \cdot \nabla)V + (\Upsilon(V) \cdot \nabla)\nabla \wedge \overline{U} = \mathbb{I}_\omega G, \tag{1.8}$$

where Υ is the inverse div-curl operator defined in (1.3).

1.4 Explicit representations of $\mathscr{F}_v$ and $\mathscr{F}_t$ for a class of regions ω

Our proofs of Theorem 1.1 and Corollary 1.2 provide explicit control spaces for a particular class of control regions. Let $\mathbb{T}^2 \setminus \omega$ be simply-connected and contain $p + [0, L]^2$ for some $p \in \mathbb{T}^2$ and $L > 0$. As the control region ω is open, it can be assumed that $2\pi/L \notin \mathbb{N}$. Following the explanations in Remarks 3.6 and 4.4 which are given later during the proofs, our constructions of $\mathscr{F}_v$ and $\mathscr{F}_t$ can be made entirely explicit in terms of closed formulas. More specifically, the space $\mathscr{F}_v$ of smooth functions $\mathbb{T}^2 \longrightarrow \mathbb{R}^2$ is spanned by

$$\begin{gathered}
\Lambda, \quad (e_k \cdot \nabla)\nabla^\perp[\chi s_l(\cdot - S_j)], \qquad \nabla^\perp[\chi s_l(\cdot - S_j)], \qquad \nabla^\perp[\chi c_l(\cdot - S_j)],\\
\Sigma, \quad (e_k \cdot \nabla)\nabla^\perp[\chi c_l(\cdot - S_j)], \quad \Delta\nabla^\perp[\chi s_l(\cdot - S_j)], \quad \Delta\nabla^\perp[\chi c_l(\cdot - S_j)],\\
(\nabla^\perp[\chi s_k(\cdot - S_j)] \cdot \nabla)\nabla^\perp[\chi s_l(\cdot - S_n)] + (\nabla^\perp[\chi s_l(\cdot - S_n)] \cdot \nabla)\nabla^\perp[\chi s_k(\cdot - S_j)],\\
(\nabla^\perp[\chi s_k(\cdot - S_j)] \cdot \nabla)\nabla^\perp[\chi c_l(\cdot - S_n)] + (\nabla^\perp[\chi c_l(\cdot - S_n)] \cdot \nabla)\nabla^\perp[\chi s_k(\cdot - S_j)],\\
(\nabla^\perp[\chi c_k(\cdot - S_j)] \cdot \nabla)\nabla^\perp[\chi c_l(\cdot - S_n)] + (\nabla^\perp[\chi c_l(\cdot - S_n)] \cdot \nabla)\nabla^\perp[\chi c_k(\cdot - S_j)]
\end{gathered}$$

with indices $k, l \in \{1, 2\}$ and $j, n \in \{1, \ldots, M\}$, while the space $\mathscr{F}_t$ of smooth zero average functions $\mathbb{T}^2 \longrightarrow \mathbb{R}$ is spanned by

$$\begin{aligned}
&\mu s_l(\cdot - S_j) - \mu \frac{\int_{\mathbb{T}^2} \mu(x) s_l(x - S_j)\, dx}{\int_{\mathbb{T}^2} \mu(x)\, dx}, && \mu c_l(\cdot - S_j) - \mu \frac{\int_{\mathbb{T}^2} \mu(x) c_l(x - S_j)\, dx}{\int_{\mathbb{T}^2} \mu(x)\, dx},\\
&(\nabla^\perp[\chi s_l(\cdot - S_j)] \cdot \nabla)\mu, && (\nabla^\perp[\chi c_l(\cdot - S_j)] \cdot \nabla)\mu,\\
&(e_l \cdot \nabla)\mu,
\end{aligned}$$

with indices $l \in \{1, 2\}$ and $j \in \{1, \ldots, M\}$, and where the yet undefined objects appearing in the representations above are specified as follows.

- $s_l(x) = \sin(x_l)$ and $c_l(x) = \cos(x_l)$ for $l \in \{1, 2\}$ and $x = (x_1, x_2) \in \mathbb{T}^2$.
- M is the square of the smallest integer above $2\pi/L$; that is, $M = \lceil 2\pi/L \rceil^2$.

- The family of translation vectors $(S_i)_{i\in\{1,\dots,M\}} \in \mathbb{R}^2$ is, as detailed in Section 3, a enumeration of $\{p - (2\pi(k-1)/\sqrt{M}, 2\pi(l-1)/\sqrt{M}) \,\Big|\, k, l \in \{1, \dots, \sqrt{M}\}\}$.

- $\mu, \chi \in C^\infty(\mathbb{T}^2;\mathbb{R})$ are as defined in Section 3.1, solely depending on ω and satisfying $\operatorname{supp}(\mu) \cup \operatorname{supp}(\chi) \subset$ ω. See Example 3.1 for a concrete choice.

- The profiles $\Lambda, \Sigma \in C^\infty(\mathbb{T}^2;\mathbb{R}^2)$ are curl-free, have linearly independent averages, and their support is contained in ω (see [41] for an explicit construction).

- $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

The space $\mathscr{F}_v$ chosen above is at most $(2 + 18\lceil 2\pi/L\rceil^2 + 8\lceil 2\pi/L\rceil^4)$-dimensional and $\mathscr{F}_t$ is at most $(2 + 8\lceil 2\pi/L\rceil^2)$-dimensional. Adding another dimension (for instance, $\operatorname{span}\mu$) to $\mathscr{F}_t$ would allow to control also the temperature average.

Remark 1.5. Utilizing Λ and Σ, which have linearly independent averages, one can adapt the explanations in Remark 4.7 to control (1.1) between states of different average without having to add more dimensions to $\mathscr{F}_v$.

Remark 1.6. When ω is arbitrary, our constructions are explicit up to Lemma 3.2, which is proved in [24, Lemma 5.1] by a contradiction argument.

1.5 Literature

The approximate controllability of Navier–Stokes and Euler systems driven by finite-dimensional but not physically localized controls has been established for the 2D periodic setting via geometric control techniques in [2, 3]. This nonlinear approach is known as the Agrachev–Sarychev method. In these works, further considered notions are the controllability of Galerkin approximations and the controllability in finite-dimensional projections (see also [4]); for an earlier and different result on the controllability of Galerkin approximations, we refer to [34]. Refinements and extensions (*e.g.*, to 3D) of the Agrachev–Sarychev method have been developed subsequently; for instance, in the articles [38, 42, 49, 50]. These ideas principally extend to other domains, provided that certain saturation properties can be verified. However, this has been done only for special configurations; for example, when orthonormal bases of trigonometric functions or spherical harmonics are available. See also [4, 47, 49, 50], and further [43, 46] for 2D and 3D rectangular domains under imposition of a slip boundary condition. We also mention that an illustration of the Agrachev–Sarychev method for the example of a 1D Burgers equation is provided by [52], and that Lagrangian and trajectorial controllability have been studied in [37] for finite-dimensional controls which may act only through few components of the Navier–Stokes system posed on the 3D torus. A negative result regarding the exact controllability of an incompressible Euler problem driven by finite-dimensional controls is obtained in [51] by a comparison argument for the Kolmogorov ε-entropy, underscoring that approximate controllability is an appropriate notion in this context. Let us also point to several applications of the Agrachev-Sarychev approach to Schrödinger equations, *e.g.*, in [16, 17, 48]. Further, we emphasize that controllability via finite-dimensional controls is related to the study of ergodicity and other properties of systems driven by degenerate noise; *e.g.*, see [27, 36] and the recent works [5, 22, 29, 39], noting that [39] treats Fourier-localized noise multiplied by a cutoff supported in an arbitrary region of the physical space.

Recently, physically localized controls of a specific degenerate structure have been obtained in [41] for the 2D incompressible Navier–Stokes system with periodic boundary conditions. The there-described approach, which already refers to the notion of observable families and the return method, subsequently inspired a first approximate (*i.e.*, global) controllability result for the Boussinesq system driven only by a temperature control [40]. In both references, the controls are not finite-dimensional but admit explicit representations involving only a finite number of control coefficients. Moreover, in [41], the control region must contain two cuts C_1 and C_2 rendering $\mathbb{T}^2 \setminus (C_1 \cup C_1)$ simply-connected, while in [40] the control region must contain a strip that cuts the torus into two doubly-connected pieces. Contrasting these works, we obtain now truly finite-dimensional controls that are physically localized. Furthermore, we do not impose any restriction on the nonempty open subset of $\mathbb{T}^2$ containing the support of the controls. To this end, our strategy relies on several new elements. For example, physically localized controls are not obtained like in [40, 41] by composing low-dimensional forces with certain flow maps that spoil their finite-dimensional nature. Instead, we introduce the tailored transport problem (1.6), which can be controlled by a finite-dimensional and physically localized force that is patched together from shifted versions of a frequency-localized control; the existence of such convection problems that are in addition compatible with the return method is an essential new ingredient. To achieve the desired effect, we carefully construct the convection profile $\overline{U}$ in (1.6), serving as a return method trajectory and encoding a combination of several geometric properties preventing it from being constant with respect to the space variables. This highlights another difference to the aforementioned studies, which rely on spatially constant return method trajectories, for instance, to handle a stretching term as in (1.8) or to drive the Boussinesq system only through the temperature.

Regarding less degenerate physically localized distributed- or boundary controls for the Navier–Stokes system and related models, there is a vast body of literature and many questions are actively studied. Attention is often paid not only to approximate- but also to exact controllability properties; for instance, exact controllability to zero (null controllability) or to trajectories. For problems without diffusion, *e.g.*, the Euler equations, exact controllability to any target in the state space might be possible (see [9, 25]). This distinction of exact controllability notions is related to the problem of understanding the reachable space; for more background in this direction, we point to the recent work [18] and the references therein. Aiming for local (small data) results, many authors have invoked linearization techniques and developed Carleman estimates for associated linear problems, leading via local inversion theorems to local exact controllability properties; for instance, [19, 23, 26], and [12, 20] for controls acting only in few components. By involving the return method, which exploits the nonlineary of the considered system, global controllability results, related to questions posed by J.-L. Lions in the 1980s-1990s (*c.f.* [33]), have been obtained for the incompressible Euler and Navier–Stokes equations [8, 9, 11, 13, 14, 25, 31, 32] and other models like the Boussinesq system and magnetohydrodynamics [6, 24, 28, 30, 44, 45], to name only a few. In these situations, the main issue is usually approximate controllability, which can be combined with a local result to achieve global exact controllability to zero or to trajectories. In particular, the study [13] resolves in both 2D and 3D a Navier slip version of J.-L. Lions' famous open problem on the approximate controllability of the Navier–Stokes system. Namely, the authors impose Navier slip-with-friction boundary conditions instead of the no-slip boundary condition; similar findings for the Boussinesq system have been obtained in [6], and [44] demonstrates approximate controllability only through the temperature for a Boussinesq system in a planar channel with thermally insulated physical boundaries on two sides along which the fluid can slip. Concerning the no-slip boundary condition, [14] establishes global controllability of the Navier–Stokes system in a rectangular region under the addition of a small phantom force (see [31] for curved boundaries). We mention also recent achievements of small-time local stabilization for the planar Navier–Stokes system in [56] and small-time global stabilization for the viscous Burgers equation with three scalar controls in [15], referring to the bibliographies of these articles for further references on stabilization problems.

2 Transport equations with generating drift

In this section, we recall a recent result from [41] on the approximate controllability of transport problems with convection along generating drifts; the argument goes back to [38] for a 3D linearized Euler problem, based on a notion of observable families from [29].

To begin with, let $\mathcal{K} \subset \mathbb{Z}^2 \setminus \{0\}$ be a finite set with $\operatorname{span}_{\mathbb{Z}}(\mathcal{K}) = \mathbb{Z}^2$ and consider the space

$$\mathscr{H} := \mathscr{H}(\mathcal{K}) := \operatorname{span}\{s_\ell, c_\ell \mid \ell \in \mathcal{K}\}, \tag{2.1}$$

where

$$s_\ell(x) := \sin(\ell \cdot x), \quad c_\ell(x) := \cos(\ell \cdot x)$$

for $x \in \mathbb{T}^2$ and $\ell \in \mathcal{K}$. Then, observable families are defined as in [38], providing a stronger version of the concept introduced in [29, Definition 4.1].

Definition 2.1. Given $T > 0$ and $N \in \mathbb{N}$, a family $(\phi_j)_{j\in\{1,\dots,N\}} \subset L^2((0,T);\mathbb{R})$ is called observable if, for all $J \subset (0,T)$, $b \in C^0(J;\mathbb{R})$, and $(a_j)_{j\in\{1,\dots,N\}} \subset C^1(J;\mathbb{R})$ it holds

$$b + \sum_{j=1}^{N} a_j \phi_j = 0 \text{ in } L^2(J;\mathbb{R}) \iff b = a_1 = \dots = a_N = 0.$$

Remark 2.2. Observable families are known to exist. For instance, one can construct them using the recipe from [38, Section 3.3].

We call a divergence-free vector field generating if it can be constructed in the below-described manner from an observable family.

Definition 2.3. Let $T > 0$. We say that $\overline{u} \in W^{1,2}((0,T); C^\infty(\mathbb{T}^2;\mathbb{R}^2))$ is generating, if it has the form

$$\overline{u}(x,t) = \kappa \sum_{\ell\in\mathcal{K}} \left(\psi^{\mathrm{s}}_\ell(t) s_\ell(x) \ell^\perp + \psi^{\mathrm{c}}_\ell(t) c_\ell(x) \ell^\perp\right) \tag{2.2}$$

for $(x,t) \in \mathbb{T}^2 \times [0,T]$, where

$$\psi^{\mathrm{s}}_\ell(t) := \phi(t) \int_0^t \phi^{\mathrm{s}}_\ell(r)\, dr, \quad \psi^{\mathrm{c}}_\ell(t) := \phi(t) \int_0^t \phi^{\mathrm{c}}_\ell(r)\, dr$$

and

- $(\phi_\ell^s, \phi_\ell^c)_{\ell \in \mathcal{K}} \subset L^2((0,T);\mathbb{R})$ is an observable family,
- $\phi \in C^1([0,T];\mathbb{R})$ obeys $\phi(t) = 0$ if an only if $t = T$,
- $\ell^\perp := (-\ell_2, \ell_1)$ for any $\ell = (\ell_1, \ell_2) \in \mathcal{K}$,
- $\kappa \in \mathbb{R} \setminus \{0\}$.

Remark 2.4. It is convenient for us to keep the parameter κ in Definition 2.3, despite its redundancy. In particular, given $(\phi_\ell^s, \phi_\ell^c)_{\ell \in \mathcal{K}}$, ϕ, and any $R > 0$, a compactness argument allows to ensure $\max_{(x,t)\in\mathbb{T}^2\times[0,T]} |\overline{u}(x,t)| < R$ by appropriately choosing κ.

The next result demonstrates a certain mixing effect that propagates energy from low frequency (Fourier-localized) sources to higher frequencies. A proof is given in [41, Theorem 2.6].

Lemma 2.5. *Let $T > 0$ and $\overline{u} \in W^{1,2}((0,T); C^\infty(\mathbb{T}^2;\mathbb{R}^2))$ be generating. Given $m \in \mathbb{N}$, $v_1 \in H^m$, and $\varepsilon > 0$, there exists a control $\zeta \in L^2((0,T);\mathscr{H})$ such that the solution to the transport problem $\partial_t v + (\overline{u}\cdot\nabla)v = \zeta$ with initial condition $v(\cdot,0) = 0$ satisfies $\|v(\cdot,T) - v_1\|_m < \varepsilon$.*

3 Finite-dimensional and physically localized transport controls

The goal of this section is to prove approximate controllability of a transport equation driven by physically localized and finite-dimensional controls. This will be possible for a special convection profile $\overline{U}$, which is a time-periodic solution to a degenerately forced Euler problem. After several preliminary constructions in Sections 3.1 and 3.2, the definition of $\overline{U}$ is given in Section 3.3; see (3.18). The main controllability result of this section is then stated in Theorem 3.13.

Notation. Let us recall that Φ^v, as defined through (1.2), refers to the flow of a sufficiently regular vector field v.

3.1 Finite-dimensional flushing profile $\overline{y}$

We turn now to the construction of a non-stationary vector field $\overline{y}$, for which $t \mapsto \overline{y}(\cdot,t)$ is a curve in a finite-dimensional subspace of $C^\infty(\mathbb{T}^2;\mathbb{R}^2)$, and such that, among others, the following properties hold for all t (see Theorem 3.4):

- $\overline{y}(\cdot,t)$ is divergence-free;
- $\overline{y}(\cdot,t)$ is equal to a gradient in the complement of ω;
- $\overline{y}$ induces a flow that flushes information in a specific way through ω.

3.1.1 Partition of unity

For a given length $L = L_\omega > 0$, we denote by $(\mathcal{O}_i^L)_{i\in\{1,\dots,M^L\}}$ an open covering of $\mathbb{T}^2$ by overlapping squares of side-length L and with bottom left corners $(o_i^L)_{i\in\{1,\dots,M^L\}} \subset \mathbb{T}^2$. Moreover, $(\mu_i^L)_{i\in\{1,\dots,M^L\}} \subset C^\infty(\mathbb{T}^2;[0,1])$ is a subordinate partition of unity:

$$\forall i \in \{1,\dots,M^L\}\colon\ \operatorname{supp}(\mu_i^L) \subset \mathcal{O}_i, \quad \sum_{i=1}^{M^L} \mu_i^L = 1.$$

As demonstrated by Example 3.1 below, these choices can be made in agreement with the following additional properties.

- There exists $\mathcal{O}^L \subset \mathbb{T}^2$ with $\overline{\mathcal{O}^L} \subset \omega$ so that $\mathcal{O}^L = \mathcal{O}_i^L + S_i^L := \{x + S_i^L \mid x \in \mathcal{O}_i^L\}$ for translation vectors $S_i^L \in \mathbb{R}^2$ and $i \in \{1,\dots,M^L\}$.
- There exists $\mu^L \in C^\infty(\mathbb{T}^2;[0,1])$ with $\mu_i^L(\cdot) = \mu^L(\cdot + S_i^L)$ for each $i \in \{1,\dots,M^L\}$.

We then introduce a cutoff $\chi^L \in C^\infty(\mathbb{T}^2;\mathbb{R})$ which satisfies $\operatorname{supp}(\chi^L) \subset \omega$ and $\chi^L = 1$ on a neighborhood of $\overline{\mathcal{O}^L}$. Furthermore, the reference time interval $[0,1]$ is partitioned with equidistant spacing

$$T^{\star,L} := 1/(3M^L + 2)$$

by means of

$$0 < t_c^{0,L} < t_a^{1,L} < t_b^{1,L} < t_c^{1,L} < \cdots < t_a^{M^L,L} < t_b^{M^L,L} < t_c^{M^L,L} < 1.$$

Simplified notations. Once the dependence on $L = L_\omega$ is clear, we drop the superscript "L" from the notations and write M, $\mathcal{O}_i$, $\mathcal{O}$, S_i, o_i, μ_i, μ, χ, t_c^0, t_a^1, t_b^1, t_c^1, $\dots$, t_a^M, t_b^M, t_c^M, $T^\star$.

The next example demonstrates that partitions of unity with the required properties exist. See also Figure 2 for a simplified illustration of the introduced setup.

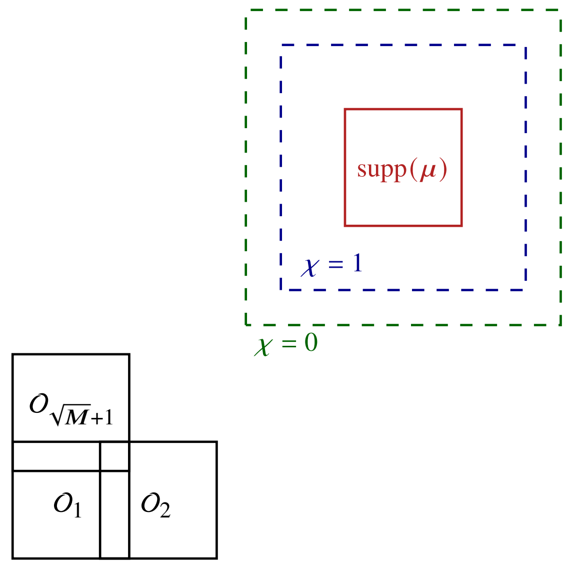

Figure 2: Illustration of the covering $(O_i)_{i\in\{1,\ldots,M\}}$ (three example squares printed) and important values of the cutoff functions μ and χ. Within the inner dashed square, which includes the support of μ, one has $\chi = 1$. Exterior to the outer dashed square, χ vanishes.

Example 3.1. Let the closure of an open square $\mathcal{O}$ with side-length $L > 0$ be contained in ω. As ω is open, we assume without loss of generality that $2\pi/L$ is not an integer; thus, $M := \lceil 2\pi/L \rceil^2 > (2\pi/L)^2$. Now, choose $\mathcal{O}_1, \ldots, \mathcal{O}_M$ as translations of $\mathcal{O}$ with overlap-width $(\sqrt{M}L - 2\pi)/\sqrt{M}$ and bottom left corners $(o_i = (o_{i,1}, o_{i,2}))_{i\in\{1,\ldots,M\}}$ given by

$$o_{i+\sqrt{M}(l-1),1} = \frac{2\pi(i-1)}{\sqrt{M}}, \quad o_{i+\sqrt{M}(l-1),2} = \frac{2\pi(l-1)}{\sqrt{M}}$$

for $i, l = 1, \ldots, \sqrt{M}$. Further, take $\widehat{\mu} \in C_0^\infty((0, L]; [0, 1])$ satisfying $\widehat{\mu}(s) = 1$ if and only if $s \in [(\sqrt{M}L - 2\pi)/(2\sqrt{M}), L]$ and define $\widetilde{\mu} \in C^\infty(\mathbb{T}; [0, 1])$ via

$$\widetilde{\mu}(s) = \mathbb{I}_{\left[0, \frac{\sqrt{M}L-2\pi}{\sqrt{M}}\right]}(s)\widehat{\mu}(s) + \mathbb{I}_{\left(\frac{\sqrt{M}L-2\pi}{\sqrt{M}}, \frac{2\pi}{\sqrt{M}}\right)}(s) + \mathbb{I}_{\left[\frac{2\pi}{\sqrt{M}}, L\right]}(s)\left(1 - \widehat{\mu}\left(s - \frac{2\pi}{\sqrt{M}}\right)\right)$$

for each $s \in \mathbb{T}$. In particular, the function $\widetilde{\mu}$ satisfies $\operatorname{supp}(\widetilde{\mu}) \subset (0, L)$ and

$$\sum_{l=1}^{\sqrt{M}} \widetilde{\mu}\left(x + \frac{2\pi(l-1)}{\sqrt{M}}\right) = 1.$$

for all $x \in \mathbb{T}$. Finally, the cutoff $\mu \in C^\infty(\mathbb{T}^2; [0,1])$ is chosen via

$$\mu(x) := \widetilde{\mu}(x_1 - o_1)\widetilde{\mu}(x_2 - o_2)$$

for $x = (x_1, x_2) \in \mathbb{T}^2$ and (o_1, o_2) being the bottom left vertex of $\mathcal{O}$. For a similar example, which however leads to a possibly larger choice of the number M, see also [41, Example 3.1].

3.1.2 Definition of the vector field $\overline{y}$

The length $L > 0$ that determines the open covering $(\mathcal{O}_i^L)_{i \in \{1,\dots,M^L\}}$ will now be fixed, together with the profile $\overline{y}$, in sole dependence on ω; see Theorem 3.4 below. Hereto, we recall first the existence of a flushing trajectory in the return method sense (see also [10, Chapter 6]). Here, we use the constructions provided by [24] for 2D and 3D flat tori.

Lemma 3.2 ([24, Lemma 5.1, Section 6])**.** *Given any $T > 0$ and a nonempty open set $\omega_0 \subset \mathbb{T}^2$, there exists a vector field $\overline{Y} \in C_0^\infty((0,T); C^\infty(\mathbb{T}^2; \mathbb{R}^2))$ that satisfies the properties*

$$\begin{gathered}
\forall (x,t) \in \mathbb{T}^2 \times [0,T]\colon \operatorname{div}(\overline{Y})(x,t) = 0,\\
\exists \Psi \in C_0^\infty((0,T); C^\infty(\mathbb{T}^2; \mathbb{R})),\ \forall (x,t) \in (\mathbb{T}^2 \setminus \omega_0) \times [0,T]\colon \overline{Y}(x,t) = \nabla\Psi(x,t),\\
\forall x \in \mathbb{T}^2,\ \exists t_x \in (0,T)\colon \boldsymbol{\Phi}^{\overline{Y}}(x,0,t_x) \in \omega_0.
\end{gathered}$$

Remark 3.3. The first two properties of $\overline{Y}$ in Lemma 3.2 ensure that $\overline{Y}$ solves in $\mathbb{T}^2 \times [0,T]$ the controlled incompressible Euler system

$$\begin{gathered}
\partial_t \overline{Y} + (\overline{Y} \cdot \nabla)\overline{Y} + \nabla\overline{p} = \mathbb{I}_\omega \overline{\xi},\\
\operatorname{div}(\overline{Y}) = 0,\\
\overline{Y}(\cdot,0) = \overline{Y}(\cdot,1) = 0
\end{gathered}$$

for a smooth pressure $\overline{p}$ and a smooth control $\overline{\xi}$ with $\operatorname{supp}(\overline{\xi}) \subset \omega \times (0,T)$. The third property of $\overline{Y}$ in Lemma 3.2 states that information originating from any location in $\mathbb{T}^2$ is flushed along $\overline{Y}$ into the given set ω_0.

The length $L > 0$ in the definition of squares $(\mathcal{O}_i^L)_{i \in \{1,\dots,M^L\}}$ is now fixed together with a finite-dimensional return method trajectory $\overline{y}$ that has more refined properties.

Theorem 3.4. *There exist $L = L_\omega > 0$, $D_\omega \in \mathbb{N}$, a D_ω-dimensional vector space $\mathscr{H}_\omega \subset C^\infty(\mathbb{T}^2;\mathbb{R}^2)$, a neighborhood $\mathcal{N}(\mathbb{T}^2 \setminus \omega)$ of $\mathbb{T}^2 \setminus \omega$, and a vector field $\overline{y} = \overline{y}_\omega \in C_0^\infty((0,1);\mathscr{H}_\omega)$ with the properties*

$$\forall h \in \mathscr{H}_\omega,\ \forall x \in \mathbb{T}^2 \colon\ \operatorname{div}(h)(x) = 0, \tag{3.1}$$

$$\forall h \in \mathscr{H}_\omega,\ \exists \varphi_h \in C^\infty(\mathbb{T}^2;\mathbb{R}),\ \forall x \in \mathcal{N}(\mathbb{T}^2 \setminus \omega)\colon\ h(x) = \nabla\varphi_h(x), \tag{3.2}$$

$$\forall t \in [0,T^\star] \cup [1-T^\star,1] \cup [t_a^1,t_b^1] \cup \cdots \cup [t_a^M,t_b^M] \colon \overline{y}(t) = 0 \tag{3.3}$$

and

$$\begin{aligned} \overline{y}(\cdot,t_c^{i-1}+t) &= -\overline{y}(\cdot,t_c^i - t),\\ \Phi^{\overline{y}}(x,t_c^{i-1},t_c^{i-1}+t) &= \Phi^{\overline{y}}(x,t_c^{i-1},t_c^i - t), \end{aligned} \tag{3.4}$$

$$\operatorname{dist}(x,\mathcal{O}_i) < L \implies \Phi^{\overline{y}}(x,0,[t_a^i,t_b^i]) = \{x + S_i\} \tag{3.5}$$

for all $(x,t) \in \mathbb{T}^2 \times [0,3T^\star]$, $i \in \{1,\dots,M\}$, and where the objects

$$M = M^L,\ \mathcal{O} = \mathcal{O}^L,\ \mathcal{O}_i = \mathcal{O}_i^L,\ S_i = S_i^L,\ t_c^i = t_c^{i,L},\ t_a^i = t_a^{i,L},\ t_b^i = t_b^{i,L},\ T^\star = T^{\star,L}$$

are chosen in dependence on L as described at the beginning of Section 3.1.

The proof of Theorem 3.4, which is based on Lemma 3.2, is carried out after several remarks below.

Remark 3.5. The property (3.4) describes how transportation along $\overline{y}$ oscillates back and forth, thereby inducing a flow with specific periodic behavior. The property (3.5) ensures that a neighborhood of each square $\mathcal{O}_i$ is transported in time t_a^i to a rigid translation of itself contained in ω.

Remark 3.6. If $\mathbb{T}^2 \setminus \omega$ is simply-connected, one can skip the proof of Theorem 3.4 and instead use the explicit example provided by [41, Theorem 3.2] with $\mathscr{H}_\omega = \mathbb{R}^2$ being a two-dimensional space of constant functions. More specifically, for any fixed basis $\{b_1,b_2\}$ of $\mathbb{R}^2$, one can take

$$\overline{y}(t) = \overline{y}_1(t)b_1 + \overline{y}_2(t)b_2,$$

where $\overline{y}_1,\overline{y}_2\colon [0,1] \longrightarrow \mathbb{R}$ are smooth functions, depending only on the time variable, chosen such that (3.1)–(3.5) hold. In this case, a covering $(\mathcal{O}_i)_{i\in\{1,\dots,M\}}$ is obtained by fixing any $L > 0$ such that a closed square of side-length L is contained in ω. Because $\mathbb{T}^2 \setminus \omega$ is simply-connected, for each $t \in [0,1]$ one can express $\overline{y}(t)$ as the sum of a gradient and a curl-free function supported in ω. Hence, the profile $\overline{y}$ solves a controlled incompressible Euler problem. See [41] for more details.

Proof of Theorem 3.4. Let us begin with a description of the idea. We start with an appropriate reference flow from Lemma 3.2 and choose $L = L_\omega$ so small that it transports entire neighborhoods of $\mathcal{O}_1^L, \dots, \mathcal{O}_{M^L}^L$ through ω. In fact, we will concatenate several scaled copies of that flow in order to prescribe the instances of time at which the content of each square is mapped into the control region. A finite-dimensional approximation of each so-obtained flow is constructed via a time discretization argument. Everything until here, we call "Building block 1". To achieve the property (3.5), we start by reversing in time the dynamics of each square $\mathcal{O}_i$ being transported along a flow described via "Building block 1" to a respective set A_i contained inside the control region; this shows how the set A_i can be mapped to a square using a return method flow. However, the so-achieved square will be $\mathcal{O}_i$ and thus may intersect $\mathbb{T}^2 \setminus \omega$. Therefore, to contain this reversed process fully in ω, we use stream function cutoffs and local shifts to modify the reversed return method flow inside of ω while setting it zero away from ω. We will call this "Building block 2". In the end, the building blocks are glued in an appropriate manner. Hereby, to ensure (3.4), original and time-reversed versions of the building blocks are iterated.

Fixing L and a reference flow. We choose $\overline{Y}$ via Lemma 3.2 for an open ball $\omega_0 \subset \omega$ of diameter $d_0 > 0$ and a fixed time $T > 0$ (*e.g.,* $T = 1$). By the compactness of $\mathbb{T}^2$ and smoothness of $\overline{Y}$, we take $L = L_\omega > 0$ so small that for each $i \in \{1, \dots, M = M^L\}$ there exists a time $t_i \in (0, T)$, a neighborhood B_i of $\mathcal{O}_i = \mathcal{O}_i^L$, and a family of balls $(B_{i,t})_{t\in[0,T]}$ of radius $d_0/6$ satisfying

$$\begin{gathered}
\operatorname{dist}(\partial B_i, \mathcal{O}_i) > L,\\
\Phi^{\overline{Y}}(B_i, 0, t_i) \subset \omega_0,\\
\operatorname{dist}(\Phi^{\overline{Y}}(B_i, 0, t_i), \partial\omega_0) > d_0/3,\\
\Phi^{\overline{Y}}(B_i, 0, t) \subset B_{i,t}
\end{gathered}$$

for all $t \in [0, T]$. The first property specifies the location of $\mathcal{O}_i$ in the open set B_i. The second and third properties express that B_i should be flushed in time t_i sufficiently "deep" into ω_0. The last property states that $\Phi^{\overline{Y}}$ cannot tear B_i too much apart: the image of B_i under the flow must be confined at each time to a ball of radius $d_0/6$.

To simplify the presentation, let us now always assume that the index i ranges over the set $\{1, \dots, M\}$.

Building block 1. The scaled versions $\overline{Y}_i(x,t) \coloneqq r_i\overline{Y}(x, r_i t)$ with $r_i \coloneqq 2t_i/T^\star$ satisfy $\Phi^{\overline{Y}}(x,0,r_i t) = \Phi^{\overline{Y}_i}(x,0,t)$ for all $(x,t) \in \mathbb{T}^2 \times [0, T^\star/2]$. Consequentially,

$$
\begin{gathered}
\Phi^{\overline{Y}_i}(B_i, 0, T^\star/2) \subset \omega_0,\\
\operatorname{dist}(\Phi^{\overline{Y}_i}(B_i, 0, T^\star/2), \partial\omega_0) > d_0/3,\\
\Phi^{\overline{Y}_i}(B_i, 0, t) \subset B_{i, r_i t}
\end{gathered}
$$

for each $t \in [0, T^\star/2]$.

Next, we introduce finite-dimensional versions of the vector fields $\overline{Y}_1, \ldots, \overline{Y}_M$. Hereto, we fix possibly large $N_i \in \mathbb{N}$, together with

$$
\begin{gathered}
\rho_i \in C_0^\infty((0, T^\star/2) \setminus \{T^\star/2N_i, T^\star/N_i, \ldots, (N_i - 1)T^\star/2N_i\}; [0,1]),\\
\mathcal{I}_k \coloneqq [(k-1)T^\star/2N_i, kT^\star/2N_i]
\end{gathered}
$$

for $k \in \{1, \ldots, N_i\}$, such that

$$
\widetilde{Y}_i(x,t) \coloneqq \rho_i(t) \sum_{k=1}^{N_i} \mathbb{I}_{\mathcal{I}_k}(t) \overline{Y}_i(x, kT^\star/2N_i) \tag{3.6}
$$

satisfies

$$
\begin{gathered}
\Phi^{\widetilde{Y}_i}(B_i, 0, T^\star/2) \subset \omega_0,\\
\operatorname{dist}(\Phi^{\widetilde{Y}_i}(B_i, 0, T^\star/2), \partial\omega_0) > d_0/3,\\
\max_{a \in \mathbb{T}^2,\, |s-r| \leqslant T^\star/2N_i} |\Phi^{\widetilde{Y}_i}(a, s, r) - a| < d_0/6
\end{gathered} \tag{3.7}
$$

and

$$
\Phi^{\widetilde{Y}_i}(B_i, 0, t) \subset B_{i, r_i t} \tag{3.8}
$$

for all $(x,t) \in \mathbb{T}^2 \times [0, T^\star/2]$.

The possibility to choose such numbers $N_1, \ldots, N_M$ and profiles $\rho_1, \ldots, \rho_M$ is due to the definition of flows of vector fields in (1.2) and Grönwall's inequality. Indeed, it holds

$$
\max_{(x,t) \in \mathbb{T}^2 \times [0, T^\star/2]} |\Phi^{\overline{Y}_i}(x,0,t) - \Phi^{\widetilde{Y}_i}(x,0,t)| \leqslant C \|\overline{Y}_i - \widetilde{Y}_i\|_{L^1((0,T^\star/2); C^0(\mathbb{T}^2;\mathbb{R}^2))},
$$

where

$$
0 < C \leqslant \max_{i \in \{1,\ldots,M\}} \mathrm{e}^{\int_0^{T^\star/2} \|\overline{Y}_i(\cdot,s)\|_{C^1(\mathbb{T}^2;\mathbb{R}^2)}\, ds}.
$$

Furthermore, the Lipschitz continuity of the smooth vector field $\overline{Y}_i$ implies

$$
\begin{aligned}
&\|\overline{Y}_i - \widetilde{Y}_i\|_{L^1((0,T^\star/2);C^0(\mathbb{T}^2;\mathbb{R}^2))} \\
&\quad \leqslant \sum_{k=1}^{N_i} \int_{\mathcal{I}_k} \left(|1-\rho(t)| \sup_{x\in\mathbb{T}^2} |\overline{Y}_i(x,t)| + \sup_{x\in\mathbb{T}^2} |\overline{Y}_i(x,t) - \overline{Y}_i(x, kT^\star/2N_i)| \right) dt \\
&\quad \leqslant C\|\rho_i - 1\|_{L^1((0,T^\star/2);\mathbb{R})} + CN_i^{-1},
\end{aligned}
$$

where $C > 0$ denotes a generic constant that can depend on ω but is independent of all data in Theorem 1.1 and Corollary 1.2. Thus, to approximate $\Phi^{\overline{Y}_i}$ by $\Phi^{\widetilde{Y}_i}$ it suffices to take large N_i, while ensuring that $\|\rho_i - 1\|_{L^1((0,T^\star/2);\mathbb{R})}$ is small. Hereby, the latter smallness is always attained by some smooth ρ_i vanishing on a neighborhood of $\{0, T^\star/2N_i, T^\star/N_i, \dots, T^\star/2\}$.

Inspecting (3.6), one finds that each $\widetilde{Y}_i$ belongs to $C_0^\infty((0,T^\star/2);\widetilde{\mathscr{H}_\omega})$ for a universal space $\widetilde{\mathscr{H}_\omega} \subset C^\infty(\mathbb{T}^2;\mathbb{R}^2)$ that consists of divergence-free functions and has at most dimension $\widetilde{N}_\omega \coloneqq \sum_{i=1}^M N_i$.

Building block 2. All elements of $\widetilde{\mathscr{H}_\omega}$ are divergence-free and ω is without loss of generality simply-connected. Thus, given any $S \in \mathbb{R}^2$, one has the stream function representations $\widetilde{Y}_i(x - S, t) = \nabla^\perp \widetilde{\Psi}_{i,S}(x,t)$ with $\widetilde{\Psi}_{i,S} \in C_0^\infty((0,T^\star/2);\widetilde{\mathscr{H}_{\omega,S}})$ for all $(x,t) \in \omega \times [0,T^\star/2]$ and an at most $\widetilde{N}_\omega$-dimensional vector space $\widetilde{\mathscr{H}_{\omega,S}} \subset C^\infty(\mathbb{T}^2;\mathbb{R})$.

Now, let $\chi_0 \in C^\infty(\mathbb{T}^2;[0,1])$ be a cutoff with $\operatorname{supp}(\chi_0) \subset \omega$ and $\chi_0 = 1$ on a neighborhood of $\overline{\omega}_0$. Then, we define for $(x,t) \in \mathbb{T}^2 \times [0,T^\star/2]$ the following time-reversed, localized, and shifted version of $\widetilde{Y}_i$:

$$
\begin{aligned}
\widehat{Y}_i(x,t) \coloneqq & - \sum_{k=1}^{N_i} \mathbb{I}_k(T^\star/2 - t) \nabla^\perp [\chi_0 \widetilde{\Psi}_{i,s_i^k}(\cdot, T^\star/2 - t)](x) \\
& + \nabla^\perp [\chi_0(x)(\widetilde{s}_i^1(T^\star/2 - t)x_1 + \widetilde{s}_i^2(T^\star/2 - t)x_2)],
\end{aligned}
$$

where the parameters $s_i^1, \dots, s_i^{N_i} \in \mathbb{R}^2$ and $\widetilde{s}_i^1, \widetilde{s}_i^1 \in C^\infty((0,T^\star/2);\mathbb{R})$ are chosen such that

$$
\begin{aligned}
\Phi^{\widehat{Y}_i}(\Phi^{\widetilde{Y}_i}(B_i, 0, T^\star/2), 0, t) &\subset \omega_0, \\
\Phi^{\widehat{Y}_i}(\Phi^{\widetilde{Y}_i}(x, 0, T^\star/2), 0, T^\star/2) &= x + S_i
\end{aligned}
$$

for $t \in [0,T^\star/2]$ and $x \in B_i$. The cutoff χ_0 ensures that the flow is stationary away from ω and that $\widehat{Y}_i$ is a gradient in $\mathbb{T}^2 \setminus \omega$.

These choices of $s_i^1, \dots, s_i^{N_i} \in \mathbb{R}^2$ and $\widetilde{s}_i^1, \widetilde{s}_i^2 \in C^\infty((0,T^\star/2);\mathbb{R})$ are possible by the following reasoning.

1) Due to (3.7), for each $k \in \{1,\dots,N_i\}$, information cannot move distances larger than $d_0/6$ along $\widetilde{Y}_i$ on the time interval $\mathcal{I}_k$. Moreover, the function ρ is known to vanish on intervals

$$[0,r_0], \quad [T^\star/2N_i - r_1, T^\star/2N_i + r_1], \quad \dots, \quad [T^\star/2 - r_{N_i}, T^\star/2]$$

for sufficiently small $r_0, r_1, \dots, r_{N_i} > 0$. Therefore, after transporting the set $\Phi^{\widetilde{Y}_i}(B_i, 0, T^\star/2)$ along the vector field $-\widetilde{Y}_i(\cdot, T^\star/2 - \cdot)$ for a duration of $T^\star/2N_i - r_{N_i-1}$, the resulting set, temporarily called A_i^1, is still contained in ω_0 and of diameter less than $d_0/6$. Hence, one can take $s_i^{N_i} = 0$. Then, one quickly pushes A_i^1 back inwards ω_0 by prescribing suitable nonzero values for $\widetilde{s}_i^1$ and $\widetilde{s}_i^2$ on an interval

$$(a_i^1, b_i^1) \subset (T^\star/2 - T^\star/2N_i, T^\star/2 - T^\star/2N_i + r_{N_i-1}).$$

Notably, the choice of ρ_i appearing in (3.6) ensures that $\rho_i(t) = 0$ for all $t \in [a_i^1, b_i^1]$. The values $s_i^{N_i-1} \in \mathbb{R}^2$ are subsequently fixed such that

$$\Phi^{\widehat{y}_i}(A_i^1, b_i^1, a_i^1) = A_i^1 + s_i^{N_i-1} = \{a + s_i^{N_i-1} \mid a \in A_i^1\} \subset \omega_0,$$

where

$$\widehat{y}_i(x,t) := \nabla^\perp[\chi_0(x)(\widetilde{s}_i^1(t)x_1 + \widetilde{s}_i^2(t)x_2)]$$

for $(x,t) \in \mathbb{T}^2 \times [0, T^\star/2]$. Hereby, the existence of such $s_i^{N_i-1}$ follows from the fact that $\widehat{y}_i$ is constant in $\overline{\omega}_0$ with respect to the space variables; its' flow rigidly translates A_i^1 within ω_0. Moreover, we can assume that the values of $(\widetilde{s}_i^1, \widetilde{s}_i^2)$ on (a_i^1, b_i^1) are fixed so that

$$\begin{gathered}\operatorname{dist}(\Phi^{\widehat{y}_i}(A_i^1, b_i^1, a_i^1), \partial\omega_0) > d_0/3,\\ \forall s \in [T^\star/2 - T^\star/2N_i, T^\star/2] \colon \Phi^{\widehat{y}_i}(A_i^1, T^\star/2, s) \subset \omega_0.\end{gathered}$$

2) Starting at $t = T^\star/2N_i$, the set $A_i^1 + s_i^{N_i-1}$ is transported along the vector field $t \mapsto -\widetilde{Y}_i(\cdot - s_i^{N_i-1}, T^\star/2 - t)$ for a duration of $T^\star/2N_i - r_{N_i-2}$ to a set A_i^2. From here, the above idea is repeated iteratively until the shifts are defined on all intervals $\mathcal{I}_k$ for $k \in \{1,\dots,N_i\}$.

Dimensions. The constructions ensure that $\widetilde{Y}_i, \widehat{Y}_i \in C_0^\infty((0,T^\star/2);\mathscr{H}_\omega)$ for an at most D_ω-dimensional vector space $\mathscr{H}_\omega \subset C^\infty(\mathbb{T}^2;\mathbb{R}^2)$, where

$$D_\omega := \widetilde{N}_\omega + \left(\sum_{i=1}^{M} N_i\right)^2 + 2.$$

More precisely, the space $\mathscr{H}_\omega$ is spanned by the following functions: 1) the elements of $\widetilde{\mathscr{H}_\omega}$; 2) the functions $\nabla^\perp(\chi_0\psi)$ with $\psi \in \widetilde{\mathscr{H}_{\omega,s_l^k}}$, $k \in \{1,\dots,N_l\}$ and $l \in \{1,\dots,M\}$; 3) the two profiles $\nabla^\perp[\chi_0 x_1]$ and $\nabla^\perp[\chi_0 x_2]$.

Conclusion of the proof. A function $\overline{y}$ with the desired properties is now defined by zero on $[0,t_c^0] \cup [t_c^M,1]$, by $\widetilde{Y}_i(\cdot, t - t_c^{i-1})$ on $[t_c^{i-1}, t_c^{i-1} + T^\star/2]$, by $\widehat{Y}_i(\cdot, t - t_c^{i-1} - T^\star/2)$ on $[t_c^{i-1} + T^\star/2, t_a^i]$, by zero on $[t_a^i, t_b^i]$, by $-\widehat{Y}_i(\cdot, T^\star/2 + t_b^i - t)$ on $[t_b^i, t_b^i + T^\star/2]$, and by $-\widetilde{Y}_i(\cdot, T^\star + t_b^i - t)$ on $[t_b^i + T^\star/2, t_c^i]$. □

3.2 Modified generating vector field $\overline{u}^\star$

For the sake of explicitness, we fix now in Definition 2.3 the natural example $\mathcal{K} = \{(1,0),(0,1)\} \subset \mathbb{Z}^2 \setminus \{0\}$. Then, the space $\mathscr{H}$ from (2.1) is four-dimensional and given by

$$\mathscr{H} = \operatorname{span}_{\mathbb{R}} \{x \mapsto \sin(x_1), x \mapsto \sin(x_2), x \mapsto \cos(x_1), x \mapsto \cos(x_2)\}. \tag{3.9}$$

All choices of finite $\mathcal{K} \subset \mathbb{Z}^2 \setminus \{0\}$ with $\operatorname{span}_{\mathbb{Z}}(\mathcal{K}) = \mathbb{Z}^2$ are allowed. But, for different $\mathcal{K}$ the control spaces obtained in the end may be different, as well.

In view of Remark 2.4, by taking $|\kappa| > 0$ in (2.2) small, we select in the sense of Definition 2.3 with $T = T^\star/2$ a divergence-free generating vector field

$$\widetilde{u}^\star \colon \mathbb{T}^2 \times [0, T^\star/2] \longrightarrow \mathbb{R}^2$$

such that

$$\bigcup_{\substack{s,t \in [0,T^\star/2],\\ S \in \mathbb{R}^2}} \Phi^{\widetilde{u}^\star(\cdot - S, \cdot)}(\operatorname{supp}(\mu), s, t) \subset \mathcal{O}, \tag{3.10}$$

where μ, χ, and $\mathcal{O}$ are fixed via Theorem 3.4 as described in Section 3.1. This is possible as $\operatorname{supp}(\mu) \subset \mathcal{O}$ and $\mathcal{O}$ is open.

Remark 3.7. The choice of the parameter κ, which ensures (3.10), is universal; it depends only on ω, and thus is independent of all prescribed data in Theorem 1.1 and Corollary 1.2. The importance of this choice will become apparent in the proofs of Lemma 3.11 and Theorem 3.13.

Because the function ϕ appearing in Definition 2.3 with $T = T^\star/2$ satisfies $\phi(T^\star/2) = 0$, we have $\widetilde{u}^\star(\cdot, 0) = \widetilde{u}^\star(\cdot, T^\star/2) = 0$. This allows us now to define the profile

$$\overline{u}^\star(x,t) := \begin{cases} \widetilde{u}^\star(x,t) & \text{if } t \in [0, T^\star/2], \\ -\widetilde{u}^\star(x, T^\star - t) & \text{if } t \in [T^\star/2, T^\star], \end{cases} \tag{3.11}$$

noting that

$$\Phi^{\overline{u}^\star}(x,0,t) = \Phi^{\overline{u}^\star}(x,0,T^\star - t) \tag{3.12}$$

for all $(x,t) \in \mathbb{T}^2 \times [0,T^\star]$. Indeed, both sides in (3.12) are equal at $t = T^\star/2$ and solve the same well-posed differential equation.

The statement of Lemma 2.5 remains true when considering convection along $\overline{u}^\star$ instead of $\overline{u}$ defined via Definition 2.3.

Lemma 3.8. *Given $m \in \mathbb{N}$, $v_0, v_1 \in H^m$, and $\varepsilon > 0$, there exists a control $g^\star \in L^2((0,T^\star);\mathscr{H})$ such that the solution $v \in C^0([0,T^\star];H^m)$ to*

$$\begin{aligned} \partial_t v + (\overline{u}^\star \cdot \nabla)v &= g^\star, \\ v(\cdot,0) &= v_0 \end{aligned} \tag{3.13}$$

satisfies

$$\|v(\cdot,T^\star) - v_1\|_m < \varepsilon. \tag{3.14}$$

Proof. First, we assume $v_0 = 0$ and set $g^\star(\cdot,t) = 0$ for $t \in [0,T^\star/2]$. As a result, it holds $v(\cdot,T^\star/2) = 0$. To determine $g^\star(\cdot,t)$ for $t \in (T^\star/2,T^\star]$, we apply Lemma 2.5 with $T = T^\star/2$ and target state v_1. The general case $v_0 \neq 0$ follows from a linear superposition principle: add the uncontrolled solution $\widetilde{v}$ with initial state v_0 to a controlled solution $\widehat{v}$ with zero initial state and target state $v_1 - v_0$, noting that $\widetilde{v}(\cdot,T^\star) = v_0$ by (3.12). □

Remark 3.9. It is known, *e.g.*, as explained in [41] or [38, Proof of Theorem 2.3], that for given $\varepsilon > 0$ and a bounded subset B of H^{m+1} with $v_0, v_1 \in B$, one can choose the control in Lemma 3.8 of the form $g^\star = \mathcal{L}_\varepsilon(v_1 - v_0)$ with a bounded linear operator $\mathcal{L}_\varepsilon \colon H^m \longrightarrow L^2((0,T^\star);\mathscr{H})$, as long as (3.14) is replaced by

$$\|v(\cdot,T^\star) - v_1\|_m < \varepsilon\|v_0 - v_1\|_{m+1}.$$

Let us briefly recall the argument when $v_0 = 0$; the general case follows by superposition as in the proof of Lemma 3.8. Hereto, consider the resolving operator $\mathcal{A}$ associating with $g^\star \in L^2((0,T^\star);\mathscr{H})$ the solution to $\partial_t v + (\overline{u}^\star \cdot \nabla)v = g^\star$ with zero initial data $v(\cdot,0) = 0$. Further, denote by $\mathcal{A}_{T^\star}$ the restriction $g \mapsto (\mathcal{A}g)(T^\star) \in H^m$. In particular, due to Lemma 3.8, the range of $\mathcal{A}_{T^\star}$ is dense in H^m. Hence, by utilizing [29, Proposition 2.6], the desired operator $\mathcal{L}_\varepsilon$ can be chosen as a continuous approximate right inverse of $\mathcal{A}_{T^\star}$.

By the Helmholtz-Hodge decomposition, the vector field $\overline{u}^\star$ admits a stream function $\overline{\phi}^\star$ in $\mathbb{T}^2 \times [0, T^\star]$. Indeed, from (2.2) and (3.11) it follows that $\overline{u}^\star$ is divergence-free and has zero average. Thus,

$$\overline{u}^\star(x,t) = \nabla^\perp \overline{\phi}^\star(x,t) \tag{3.15}$$

for $(x,t) \in \mathbb{T}^2 \times [0, T^\star]$. Moreover, recall that we fixed in Section 3.1 the translation vectors $(S_i)_{i\in\{1,\dots,M\}} \subset \mathbb{R}^2$ such that

$$\mathcal{O} = \mathcal{O}_i + S_i \tag{3.16}$$

for each $i \in \{1, \dots, M\}$. Then, by the definition of $\overline{u}^\star$ in (3.11), one has analogues of (3.12) for flows arising from certain variations of the vector field

$$(x,t) \mapsto \nabla^\perp[\chi(x)\overline{\phi}^\star(x,t)],$$

where χ is the cutoff introduced in Section 3.1. For instance, it holds

$$\begin{aligned}\Phi^{\nabla^\perp[\chi(\cdot)\overline{\phi}^\star(\cdot-S,\cdot)]}(x,0,s) &= \Phi^{\nabla^\perp[\chi(\cdot)\overline{\phi}^\star(\cdot-S,\cdot)]}(x,0,T^\star - s),\\ \Phi^{\nabla^\perp[\chi(\cdot)\overline{\phi}^\star(\cdot-S_i,\cdot-t_a^i)]}(x,t_a^i,t) &= \Phi^{\nabla^\perp[\chi(\cdot)\overline{\phi}^\star(\cdot-S_i,\cdot-t_a^i)]}(x,t_a^i,t_b^i-(t-t_a^i))\end{aligned} \tag{3.17}$$

for all $i \in \{1, \dots, M\}$, $S \in \mathbb{R}^2$, $x \in \mathbb{T}^2$, $s \in [0, T^\star]$, and $t \in [t_a^i, t_b^i]$. Indeed, both sides in each line of (3.17) satisfy the same differential equations with identical states at $s = T^\star/2$ and $t = t_a^i + T^\star/2$, respectively.

The next lemma is a consequence of (3.10) for sufficiently small $|\kappa| > 0$. However, in view of Remark 2.4 and (3.15), it is shorter to argue that the lemma follows from a new (smaller) choice of $|\kappa| > 0$ which depends only on ω.

Lemma 3.10. *There exists $r > 0$ with $\chi = 1$ on the r-neighborhood $\mathcal{N}_r$ of the reference square $\mathcal{O}$ with side-length L (defined in Section 3.1) and such that $\mathcal{N}_r$ is also a neighborhood of* $\bigcup_{s,t\in[0,T^\star],\, S\in\mathbb{R}^2} \Phi^{\nabla^\perp[\chi(\cdot)\overline{\phi}^\star(\cdot-S,\cdot)]}(\operatorname{supp}(\mu), s, t)$.

3.3 Definition of $\overline{U}$ based on $\overline{y}$ and $\overline{u}^\star$

We introduce a function $\overline{U} \in W^{1,2}((0,1); C^\infty(\mathbb{T}^2;\mathbb{R}^2))$ that will be used in Section 3.4 as a convection profile for linear transport equations steered by physically localized and finite-dimensional controls. More precisely,

$$\overline{U}(x,t) := \overline{y}(x,t) + \sum_{i=1}^{M} \mathbb{I}_{[t_a^i,t_b^i]}(t)\nabla^\perp[\chi(x)\overline{\phi}^\star(x - S_i, t - t_a^i)] \tag{3.18}$$

for $(x,t) \in \mathbb{T}^2 \times [0,1]$. Aside of being divergence-free in $\mathbb{T}^2$, the vector field $\overline{U}(\cdot,t)$ is a gradient in a neighborhood of $\mathbb{T}^2 \setminus \omega$ for each $t \in [0,1]$. Moreover, due to (3.3), (3.4), and (3.17), the flow of $\overline{U}$ satisfies

$$\Phi^{\overline{U}}(x,0,t) = \Phi^{\overline{U}}(x,t_c^M,t) = \Phi^{\overline{U}}(x,t_a^l,t_b^l) = \Phi^{\overline{U}}(x,t_c^{l-1},t_c^l) = \Phi^{\overline{U}}(x,0,1) = x \tag{3.19}$$

for all $x \in \mathbb{T}^2$, $t \in [0,T^\star]$, and $l \in \{1,\dots,M\}$.

The following two lemmas provide basic properties of the considered flows related to $\overline{U}$.

Lemma 3.11. *Given $x \in \mathbb{T}^2$ and $i \in \{1,\dots,M\}$ such that $\mu(\Phi^{\overline{U}}(x,0,t_a^i+s)) \neq 0$ or $\mu(\Phi^{\overline{u}^\star(\cdot-S_i,\cdot)}(x+S_i,0,s)) \neq 0$ are satisfied for at least one $s \in [0,T^\star]$, it holds*

$$\Phi^{\overline{U}}(x,0,t_a^i+t) = \Phi^{\overline{u}^\star(\cdot-S_i,\cdot)}(x+S_i,0,t)$$

for all $t \in [0,T^\star]$.

Proof. By (3.16), (3.18), and (3.19) and Theorem 3.4, one has $\Phi^{\overline{U}}(z,0,t_a^i) = z+S_i$ for each z in the L-neighborhood of O_i. Moreover, from the definition of $\overline{U}$ in (3.18), one can infer

$$\Phi^{\overline{U}}(z,t_a^i+r,t_a^i+t) = \Phi^{\nabla^\perp[\chi(\cdot)\overline{\phi}^\star(\cdot-S_i,\cdot)]}(z,r,t)$$

for all $z \in \mathbb{T}^2$ and $r,t \in [0,T^\star]$.

Case 1. If there exists a number $s \in [0,T^\star]$ such that $\mu(\Phi^{\overline{U}}(x,0,t_a^i+s)) \neq 0$, by Lemma 3.10 this means that $\Phi^{\overline{U}}(x,0,t_a^i)$ lies in a L-neighborhood of O. As a consequence of Theorem 3.4, the point x then belongs to an L-neighborhood of O_i, which yields $\Phi^{\nabla^\perp[\chi(\cdot)\overline{\phi}^\star(\cdot-S_i,\cdot)]}(x+S_i,0,s) \in \operatorname{supp}(\mu)$ because of

$$\begin{aligned}
\operatorname{supp}(\mu) \ni \Phi^{\overline{U}}&(x,0,t_a^i+s)\\
&= \Phi^{\overline{U}}(\Phi^{\overline{U}}(x,0,t_a^i),t_a^i,t_a^i+s)\\
&= \Phi^{\overline{U}}(x+S_i,t_a^i,t_a^i+s)\\
&= \Phi^{\nabla^\perp[\chi(\cdot)\overline{\phi}^\star(\cdot-S_i,\cdot)]}(x+S_i,0,s).
\end{aligned}$$

Therefore, it follows from Lemma 3.10 that $\chi(\Phi^{\nabla^\perp[\chi(\cdot)\overline{\phi}^\star(\cdot-S_i,\cdot)]}(x+S_i,0,t)) = 1$ for all $t \in [0,T^\star]$. As a result, the well-posed problem $\Phi'(t) = \overline{u}^\star(\Phi(t)-S_i,t)$ with $\Phi(0) = x+S_i$ is satisfied on $[0,T^\star]$ by $t \mapsto \Phi^{\overline{u}^\star(\cdot-S_i,\cdot)}(x+S_i,0,t)$ and also by $t \mapsto \Phi^{\nabla^\perp[\chi(\cdot)\overline{\phi}^\star(\cdot-S_i,\cdot)]}(x+S_i,0,t)$; thus, these two functions are equal.

Case 2. If there exists $s \in [0, T^\star]$ such that $\mu(\Phi^{\overline{u}^\star(\cdot - S_i, \cdot)}(x + S_i, 0, s)) \neq 0$, then (3.10) and (3.11) ensure that $\chi(z) = 1$ for all z from a neighborhood of $\cup_{t \in [0,T^\star]} \Phi^{\overline{u}^\star(\cdot - S_i, \cdot)}(x + S_i, 0, t)$. This implies

$$\Phi^{\overline{u}^\star(\cdot - S_i, \cdot)}(x + S_i, 0, t) = \Phi^{\nabla^\perp[\chi(\cdot)\overline{\phi}^\star(\cdot - S_i, \cdot)]}(x + S_i, 0, t)$$

for $t \in [0, T^\star]$. Hence, the assertion can be concluded by analysis similar to the previous case. ◻

Lemma 3.12. *Let $T > 0$ and $v\colon \mathbb{T}^2 \times [0, T] \longrightarrow \mathbb{R}^2$ sufficiently regular such that the flow Φ^v is well-defined. Then, one has*

$$\Phi^{v(\cdot - S, \cdot)}(x + S, 0, t) = \Phi^v(x, 0, t) + S$$

for all $S \in \mathbb{R}^2$ and $(x, t) \in \mathbb{T}^2 \times [0, T]$.

3.4 Controllability of convection along $\overline{U}$

The goal of this section is to demonstrate approximate controllability of the linear transport equation

$$\partial_t V + (\overline{U} \cdot \nabla)V = G,$$

where $\overline{U}$ is defined via (3.18) and G denotes a finite-dimensional physically localized control.

Theorem 3.13. *There exists a finite-dimensional space $\mathscr{F}_t$ consisting of smooth zero average functions $\mathbb{T}^2 \longrightarrow \mathbb{R}$ supported in ω such that the following statement holds. Given any $m \in \mathbb{N}$, $v_0, v_1 \in H^m$, and $\varepsilon > 0$, there is a control $G \in L^2((0, 1); \mathscr{F}_t)$ such that the solution $V \in C^0([0, 1]; H^m)$ to*

$$\begin{aligned} \partial_t V + (\overline{U} \cdot \nabla)V &= G, \\ V(\cdot, 0) &= v_0 \end{aligned} \tag{3.20}$$

satisfies

$$\|V(\cdot, 1) - v_1\|_m < \varepsilon. \tag{3.21}$$

Proof. The first step is to define a finite-dimensional and physically localized auxiliary control, which however fails to be of zero average. In the second step, the average is corrected. The last step will summarize how the universal space $\mathscr{F}_t$ arises from the foregoing constructions.

Step 1. Auxiliary control. Let the functions $v \in C^0([0,T^\star];H^m(\mathbb{T}^2;\mathbb{R}))$ and $g^\star \in L^2((0,T^\star);\mathscr{H})$ be obtained by Lemma 3.8 such that v solves the transport equation (3.13) with control $g^\star$ and satisfies

$$\|v(\cdot,T^\star) - v_1\|_m < \varepsilon. \tag{3.22}$$

Then, we define for all $(x,t) \in \mathbb{T}^2 \times [0,1]$ the auxiliary control

$$\widetilde{G}(x,t) \coloneqq \mu(x) \sum_{i=1}^{M} \mathbb{I}_{[t_a^i, t_b^i]}(t) g^\star(x - S_i, t - t_a^i), \tag{3.23}$$

where μ with $\operatorname{supp}(\mu) \subset \mathcal{O} \subset \omega$ is the cutoff from Section 3.1 and $S_1,\dots,S_M$ satisfy (3.16).

Associated with the control $\widetilde{G}$, let $\widetilde{V}$ be the solution to $\partial_t \widetilde{V} + (\overline{U}\cdot\nabla)\widetilde{V} = \widetilde{G}$ with initial condition $\widetilde{V}(\cdot,0) = v_0$. By the method of characteristics and Duhamel's principle, it holds

$$\widetilde{V}(x,t) = v_0(\Phi^{\overline{U}}(x,t,0)) + \int_0^t \widetilde{G}(\Phi^{\overline{U}}(x,t,s),s)\,ds$$

for all $(x,t) \in \mathbb{T}^2 \times [0,1]$. Now, let $x \in \mathbb{T}^2$ be arbitrary and note that $\Phi^{\overline{U}}(x,1,0) = x$ holds due to (3.19). Therefore,

$$\begin{aligned}\widetilde{V}(x,1) &= v_0(x) + \int_0^1 \widetilde{G}(\Phi^{\overline{U}}(x,1,s),s)\,ds\\ &= v_0(x) + \sum_{i=1}^{M}\int_0^1 \mu(\Phi^{\overline{U}}(x,1,s))\mathbb{I}_{[t_a^i,t_b^i]}(s) g^\star(\Phi^{\overline{U}}(x,1,s) - S_i, s - t_a^i)\,ds.\end{aligned}$$

Again by (3.19), it follows that

$$\begin{aligned}\widetilde{V}(x,1) &= v_0(x) + \sum_{i=1}^{M}\int_0^1 \mu(\Phi^{\overline{U}}(x,0,s))\mathbb{I}_{[t_a^i,t_b^i]}(s) g^\star(\Phi^{\overline{U}}(x,0,s) - S_i, s - t_a^i)\,ds\\ &= v_0(x) + \sum_{i=1}^{M}\int_{t_a^i}^{t_b^i} \mu(\Phi^{\overline{U}}(x,0,s)) g^\star(\Phi^{\overline{U}}(x,0,s) - S_i, s - t_a^i)\,ds,\end{aligned}$$

which is due to a change of variables under the integral sign equal to

$$v_0(x) + \sum_{i=1}^{M}\int_0^{T^\star} \mu(\Phi^{\overline{U}}(x,0,s+t_a^i)) g^\star(\Phi^{\overline{U}}(x,0,s+t_a^i) - S_i, s)\,ds.$$

In view of Lemma 3.11, the previous expression equals

$$v_0(x) + \sum_{i=1}^{M} \int_0^{T^\star} \mu(\Phi^{\overline{u}^\star(\cdot - S_i, \cdot)}(x+S_i, 0, s)) g^\star(\Phi^{\overline{u}^\star(\cdot - S_i, \cdot)}(x+S_i, 0, s) - S_i, s)\, ds,$$

and therefore Lemma 3.12 allows to infer that

$$\widetilde{V}(x,1) = v_0(x) + \sum_{i=1}^{M} \int_0^{T^\star} \mu(\Phi^{\overline{u}^\star}(x,0,s) + S_i) g^\star(\Phi^{\overline{u}^\star}(x,0,s) + S_i - S_i, s)\, ds.$$

Recalling that $(\mu_i)_{i\in\{1,\dots,M\}}$ is the partition of unity from Section 3.1 and that $\overline{u}^\star$ satisfies (3.12), one obtains

$$\begin{aligned} \widetilde{V}(x,1) &= v_0(x) + \sum_{i=1}^{M} \int_0^{T^\star} \mu_i(\Phi^{\overline{u}^\star}(x,0,s)) g^\star(\Phi^{\overline{u}^\star}(x,0,s), s)\, ds \\ &= v_0(\Phi^{\overline{u}^\star}(x, T^\star, 0)) + \int_0^{T^\star} g^\star(\Phi^{\overline{u}^\star}(x, T^\star, s), s)\, ds. \end{aligned}$$

This demonstrates that $\widetilde{V}(x,1) = v(x,T^\star)$, and because v satisfies (3.22), one arrives at

$$\|\widetilde{V}(\cdot,1) - v_1\|_m = \|v(\cdot,T^\star) - v_1\|_m < \varepsilon.$$

Step 2. Control with zero average. It remains to define suitable modifications of $\widetilde{V}$ and $\widetilde{G}$ with zero average. Since $\widetilde{V}(\cdot,1) = v(\cdot,T^\star)$ has zero average because v is the function from Lemma 3.8, we introduce

$$V(x,t) \coloneqq \widetilde{V}(x,t) - \frac{\mu(x)\int_{\mathbb{T}^2} \widetilde{V}(x,t)\, dx}{\int_{\mathbb{T}^2} \mu(x)\, dx}$$

which by construction satisfies together with the modified control

$$G(x,t) \coloneqq \widetilde{G}(x,t) - \frac{\frac{d}{dt}\int_{\mathbb{T}^2} \widetilde{V}(z,t)\, dz}{\int_{\mathbb{T}^2} \mu(z)\, dz}\mu(x) - \frac{\int_{\mathbb{T}^2} \widetilde{V}(z,t)\, dz}{\int_{\mathbb{T}^2} \mu(z)\, dz}(\overline{U}(x,t)\cdot\nabla)\mu(x) \quad (3.24)$$

the transport equation (3.20) and the condition (3.21).

Step 3. The space $\mathscr{F}_t$. By Definition 2.3, Theorem 3.4, (3.9), (3.11), and (3.18), there exist smooth universal functions $U_1, \dots, U_{4M+D_\omega} \in C^\infty(\mathbb{T}^2; \mathbb{R}^2)$ such that

$$\overline{U}(x,t) = \sum_{i=1}^{4M+D_\omega} u_i(t) U_i(x) \quad (3.25)$$

for coefficients $u_1, \dots, u_{4M+D_\omega} \in L^2((0,1);\mathbb{R})$. More specifically, one has $\overline{y} \in C_0^\infty((0,1);\mathscr{H}_\omega)$ for an at most D_ω-dimensional space $\mathscr{H}_\omega \subset C^\infty(\mathbb{T}^2;\mathbb{R}^2)$; hence, we can choose $U_1, \dots, U_{D_\omega} \in C^\infty(\mathbb{T}^2;\mathbb{R}^2)$ so that $\overline{y}(x,t) = \sum_{i=1}^{D_\omega} u_i(t)U_i(x)$. Further, by Definition 2.3, there are universal profiles $\Phi_1, \dots, \Phi_4 \in C^\infty(\mathbb{T}^2;\mathbb{R}^2)$ such that the stream function $\overline{\phi}^\star$ in (3.18) has the four-dimensional representation

$$\overline{\phi}^\star(x,t) = \phi_1(t)\Phi_1(x) + \dots + \phi_4(t)\Phi_4(x),$$

with coefficients $\phi_1, \dots, \phi_4 \in L^2((0,1);\mathbb{R})$. This provides

$$U_{D_\omega+1} = \nabla^\perp[\chi\Phi_1(\cdot - S_1)], \dots, U_{D_\omega+4} = \nabla^\perp[\chi\Phi_4(\cdot - S_1)], \\ \dots, U_{D_\omega+4M} = \nabla^\perp[\chi\Phi_4(\cdot - S_M)].$$

In a similar manner, from (3.9) it follows that there are universal functions $\widetilde{G}_1, \dots, \widetilde{G}_{4M} \in C^\infty(\mathbb{T}^2;\mathbb{R}^2)$ with $\operatorname{supp}(\widetilde{G}_i) \subset \omega$ for $i \in \{1, \dots, 4M\}$ such that $\widetilde{G}$ from (3.23) is with coefficients $\widetilde{g}_1, \dots, \widetilde{g}_{4M} \in L^2((0,1);\mathbb{R})$ of the form

$$\widetilde{G}(x,t) = \sum_{i=1}^{4M} \widetilde{g}_i(t)\widetilde{G}_i(x).$$

In order to replace $\widetilde{G}_1, \dots, \widetilde{G}_{4M}$ with zero average versions, we first expand $\widetilde{G}$ in the following way

$$\widetilde{G}(x,t) = \sum_{i=1}^{4M} \widetilde{g}_i(t)\widehat{G}_i(x) + \sum_{i=1}^{4M} \frac{\widetilde{g}_i(t)\int_{\mathbb{T}^2} \widetilde{G}_i(z)\,dz}{\int_{\mathbb{T}^2} \mu(z)\,dz}\mu(x),$$

where

$$\widehat{G}_i(x) \coloneqq \widetilde{G}_i(x) - \frac{\int_{\mathbb{T}^2} \widetilde{G}_i(z)\,dz}{\int_{\mathbb{T}^2} \mu(z)\,dz}\mu(x)$$

for $i \in \{1, \dots, 4M\}$. Since the right-hand side in (3.24) and its last term both have zero average, it follows that

$$\sum_{i=1}^{4M} \frac{\widetilde{g}_i(t)\int_{\mathbb{T}^2} \widetilde{G}_i(z)\,dz}{\int_{\mathbb{T}^2} \mu(z)\,dz}\mu(x) = \frac{\frac{d}{dt}\int_{\mathbb{T}^2} \widetilde{V}(z,t)\,dz}{\int_{\mathbb{T}^2} \mu(z)\,dz}\mu(x).$$

Plugging this into (3.24), we obtain $G_1, \dots, G_{8M+D_\omega} \in C^\infty(\mathbb{T}^2;\mathbb{R}^2)$ with zero average and $\operatorname{supp}(G_i) \subset \omega$ for $i \in \{1, \dots, 8M + D_\omega\}$ such that

$$G(x,t) = \sum_{i=1}^{8M+D_\omega} g_i(t)G_i(x)$$

for coefficients $g_1, \dots, g_{8M+D_\omega} \in L^2((0,1);\mathbb{R})$. Thus, the space $\mathscr{F}_t$ can be taken as the span of $G_1, \dots, G_{8M+D_\omega}$. □

Remark 3.14. The result of Theorem 3.13 remains true for states with nonzero average, as long as one adds to $\mathscr{F}_t$ the one-dimensional space spanned by the smooth cutoff μ from Section 3.1 (or by any other smooth nonzero average function supported in ω). Indeed, if τ_0 denotes the average of v_0 and τ_1 is the average of v_1, one can take $\tau \in C_0^\infty((0,1);\mathbb{R})$ such that $\int_0^1 \tau(s)\,ds = 1$ and observe that the solution to

$$\partial_t \widetilde{v} + (\overline{U}\cdot\nabla)\widetilde{v} = \zeta := \frac{\tau(\tau_1-\tau_0)}{\int_{\mathbb{T}^2}\mu(x)\,dx}\mu$$

with initial condition $\widetilde{v}(\cdot,0) = \tau_0$ satisfies $\int_{\mathbb{T}^2}\widetilde{v}(x,1)\,dx = \tau_1$. Subsequently, one obtains a zero average control $\widehat{G}$ by applying Theorem 3.13 with initial state $v_0-\tau_0$ and target state $v_1-\widetilde{v}(\cdot,1)$. The desired control for the nonzero average trajectory is then $G = \widehat{G}+\zeta$. Notably, while this is an approximate controllability result, the average is controlled exactly.

Remark 3.15. Let $\varepsilon > 0$ and B a bounded set in H^{m+1} with $v_0, v_1 \in B$. By Remark 3.9 and the constructions in (3.23) and (3.24), the controls from Theorem 3.13 and Remark 3.14 for the modified target condition

$$\|V(\cdot,1)-v_1\|_m < \varepsilon\|v_0-v_1\|_{m+1}$$

can be chosen of the form $G = \mathcal{L}_\varepsilon(v_1-v_0)$, where $\mathcal{L}_\varepsilon$ is a continuous linear operator $H^m \longrightarrow L^2((0,1);\mathscr{F}_t)$.

The following auxiliary lemma, which will be used in Section 4, emphasizes the 1-periodicity of homogeneous transportation with stretching effect along the vector field $\overline{U}$ from (3.18).

Lemma 3.16. *For each $V_0 \in H^1$, the solution V to the homogeneous linear convection problem*

$$\partial_t V + (\overline{U}\cdot\nabla)V + (\Upsilon(V)\cdot\nabla)\nabla\wedge\overline{U} = 0$$

with $V(\cdot,0) = V_0$ satisfies $V(\cdot,1) = V_0$.

Proof. Due to the definition of $\overline{U}$ in (3.18), the problem reduces to showing that the solution V^l on $\mathbb{T}^2\times[t_c^l, t_c^{l+1}]$ to

$$\partial_t V^l + (\overline{U}\cdot\nabla)V^l + (\Upsilon(V^l)\cdot\nabla)\nabla\wedge\overline{U} = 0$$

with initial state $V^l(\cdot,t_c^l) = V_0$ satisfies $V(t_c^{l+1}) = V_0$ for each $l \in \{0,\dots,M-1\}$ and $V_0 \in H^1$. This follows from a time reversibility argument and the properties (3.4), (3.11), (3.12), (3.17), and (3.19). Indeed,

$$\overline{U}(\cdot, t_c^l + 3T^\star/2 + t) = -\overline{U}(\cdot, t_c^l + 3T^\star/2 - t)$$

holds for all $t \in [0, 3T^\star/2]$. Therefore, each $\widehat{V}^l(\cdot, t) := V^l(\cdot, t_c^l + 3T^\star/2 - t)$ satisfies in $\mathbb{T}^2 \times [0, 3T^\star/2]$ the equation

$$\partial_t \widehat{V}^l + (\overline{U}(\cdot, t_c^l + 3T^\star/2 + \cdot) \cdot \nabla)\widehat{V}^l + (\Upsilon(\widehat{V}^l) \cdot \nabla)\nabla \wedge \overline{U}(\cdot, t_c^l + 3T^\star/2 + \cdot) = 0$$

with initial condition $\widehat{V}^l(\cdot, 0) = V^l(\cdot, t_c^l + 3T^\star/2)$. Due to $\widehat{V}^l(\cdot, 3T^\star/2) = V^l(\cdot, t_c^l) = V_0$, noting that $\widetilde{V}^l(\cdot, t) := V^l(\cdot, t_c^l + 3T^\star/2 + t)$ and $\widehat{V}^l$ both solve on $[0, 3T^\star/2]$ the same well-posed problem, the claim follows. □

4 The nonlinear problem

Like the 2D incompressible Navier–Stokes equations (see [21, 54, 55]), the planar Boussinesq system (1.1) with nonzero viscosity and thermal diffusivity is well-posed in common Sobolev space settings. We will work with the vorticity-temperature formulation obtained from (1.1) by formally applying the operator $\nabla\wedge$ in the velocity equation.

Given $T > 0$ and $m \in \mathbb{N}$, we define the spaces $\mathcal{X}_T^m := \mathcal{A}_T^{m-1} \times \mathcal{A}_T^m$ with the norm $\|(w, \theta)\|_{\mathcal{X}_T^m} := \|w\|_{\mathcal{A}_T^{m-1}} + \|\theta\|_{\mathcal{A}_T^m}$, where

$$\mathcal{A}_T^k := C^0([0, T]; H^k(\mathbb{T}^2; \mathbb{R})) \cap L^2((0, T); H^{k+1}(\mathbb{T}^2; \mathbb{R}))$$

is for $k \in \mathbb{N}_0$ equipped with

$$\|\cdot\|_{\mathcal{A}_T^k} := \|\cdot\|_{C^0([0,T];H^k(\mathbb{T}^2;\mathbb{R}))} + \|\cdot\|_{L^2((0,T);H^{k+1}(\mathbb{T}^2;\mathbb{R}))}.$$

Then, for any $(w_0, \theta_0) \in H^{m-1} \times H^m$, $(h_1, h_2) \in L^2((0, T); H^{m-2} \times H^{m-1})$ and $A \in W^{1,2}((0, T); \mathbb{R}^2)$, there exists a unique solution $(w, \theta) \in \mathcal{X}_T^m$ to the Boussinesq system in vorticity-temperature form

$$\begin{aligned} \partial_t w - \nu \Delta w + (u \cdot \nabla)\, w &= \partial_1 \theta + h_1, \\ u(\cdot, t) &= \Upsilon(w, A), \\ \partial_t \theta - \tau \Delta \theta + (u \cdot \nabla)\, \theta &= h_2, \\ w(\cdot, 0) = w_0, \quad \theta(\cdot, 0) &= \theta_0. \end{aligned} \tag{4.1}$$

Moreover, the resolving operator for (4.1), associating with $(w_0, \theta_0, h_1, h_2, A)$ the solution (w, θ) to (4.1), is continuous and given by

$$\begin{aligned} S_T \colon H^{m-1} \times H^m \times L^2((0, T); H^{m-2} \times H^{m-1}) \times W^{1,2}((0, T); \mathbb{R}^2) &\longrightarrow \mathcal{X}_T^m, \\ (w_0, \theta_0, h_1, h_2, A) &\longmapsto S_T(w_0, \theta_0, h_1, h_2, A) := (w, \theta). \end{aligned}$$

4.1 Proof of Theorem 1.1

To begin with, let us emphasize that the profile $\overline{U}$ from (3.18) satisfies due to Theorem 3.4 the following controllability problem for the incompressible Euler system in $\mathbb{T}^2 \times (0,1)$:

$$\begin{gathered}\partial_t \overline{U} + (\overline{U} \cdot \nabla)\overline{U} + \nabla \overline{P} = \mathbb{I}_\omega \overline{H},\\ \operatorname{div}(\overline{U}) = 0,\\ \overline{U}(\cdot, 0) = \overline{U}(\cdot, 1) = 0,\end{gathered} \tag{4.2}$$

where

$$\begin{gathered}\overline{P} \in L^2((0,1); C^\infty(\mathbb{T}^2; \mathbb{R}^2)), \quad \overline{H} \in L^2((0,1); C^\infty(\mathbb{T}^2; \mathbb{R}^2)),\\ \operatorname{supp}(\overline{H}) \subset \omega \times (0,1).\end{gathered}$$

Moreover, since $\overline{U}(\cdot, t)$ is a gradient in a neighborhood of $\mathbb{T}^2 \setminus \omega$ at each time $t \in [0,1]$, the functions

$$H^{1,1} \coloneqq \nabla \wedge \overline{H}, \quad H^{1,2} \coloneqq -\Delta(\nabla \wedge \overline{U})$$

satisfy

$$\operatorname{supp}(H^{1,1}) \cup \operatorname{supp}(H^{1,2}) \subset \omega \times [0,1].$$

Further, we define

$$\begin{gathered}\overline{U}_\delta(\cdot, t) \coloneqq \delta^{-1}\overline{U}(\cdot, \delta^{-1}t),\\ A_\delta(t) \coloneqq \delta^{-1} \int_0^{\delta^{-1}t} \int_{\mathbb{T}^2} \overline{H}(x,s)\, dx ds,\\ H_{1,\delta}(\cdot, t) \coloneqq \delta^{-2} H^{1,1}(\cdot, \delta^{-1}t) + \delta^{-1}\nu H^{1,2}(\cdot, \delta^{-1}t)\end{gathered} \tag{4.3}$$

for $\delta > 0$ and all $t \in [0, \delta]$. As $\overline{U}$ is by construction supported in $(0,1)$ with respect to the time variable, it follows from (4.2) that

$$A_\delta(\delta) = \delta^{-1} \int_0^1 \int_{\mathbb{T}^2} \overline{H}(x,s)\, dx ds = 0.$$

The next theorem is part of the return method argument, employed here to relate the temperature in (4.1) at a small time $t = \delta$ with the solution to (3.20) evaluated at $t = 1$. The present situation is slightly different from that in [40, 41], but the proof follows closely the corresponding arguments from these references; the general approach is due to [8], and we refer also to [24, 38].

Theorem 4.1. *Assume that $m \geqslant 2$, $(w_0, \theta_0) \in H^m \times H^{m+1}(\mathbb{T}^2;\mathbb{R})$, $A \in \mathbb{R}^2$, $b \in C^\infty(\mathbb{T}^2 \times [0,1];\mathbb{R}^2)$, and $(h_1, h_2) \in L^2((0,T); H^{m-2} \times H^{m-1})$ for some $T > 0$. Moreover, denote by $(v_\delta, \vartheta_\delta)_{\delta \in (0,\min\{1,T\})}$ the solutions to the linear problems with parameter $\delta \in (0, \min\{1,T\})$:*

$$\begin{aligned} \partial_t v_\delta + (\overline{U} \cdot \nabla) v_\delta + (\Upsilon(v_\delta, B_\delta) \cdot \nabla)\nabla \wedge \overline{U} &= \partial_1 \vartheta_\delta + \nabla \wedge b, \\ \partial_t \vartheta_\delta + (\overline{U} \cdot \nabla)\vartheta_\delta &= \zeta_\delta, \\ v_\delta(\cdot, 0) = w_0, \quad \vartheta_\delta(\cdot,0) &= \delta\theta_0, \end{aligned} \tag{4.4}$$

where

$$B_\delta(t) \coloneqq A + \int_0^t \int_{\mathbb{T}^2} b(x,s)\,dxds + e_2 \int_0^t \int_{\mathbb{T}^2} \delta^{-1}\vartheta_\delta(x,s)\,dxds$$

for $t \in [0,1]$ and the family $(\zeta_\delta)_{\delta \in (0,\min\{1,T\})} \subset L^2((0,1); C^\infty(\mathbb{T}^2;\mathbb{R}))$ of forces is chosen such that

$$\sup_{t\in[0,1]} \|\vartheta_\delta(\cdot,t)\|_{m+1} = \mathscr{O}(\delta) \text{ as } \delta \longrightarrow 0. \tag{4.5}$$

Further, for $t \in [0,\delta]$, denote

$$\begin{aligned} H_{2,\delta}(\cdot,t) \coloneqq \delta^{-2}\zeta_\delta(\cdot, \delta^{-1}t), \quad b_\delta(\cdot,t) \coloneqq \delta^{-1} b(\cdot, \delta^{-1}t), \\ \aleph_\delta(t) \coloneqq A_\delta(t) + B_\delta(\delta^{-1}t). \end{aligned}$$

Then, as $\delta \longrightarrow 0$, one has in $H^{m-1} \times H^m(\mathbb{T}^2;\mathbb{R})$ the convergence

$$S_\delta(w_0, \theta_0, h_1 + H_{1,\delta} + \nabla \wedge b_\delta, h_2 + H_{2,\delta}, \aleph_\delta)|_{t=\delta} - (v_\delta, \delta^{-1}\vartheta_\delta)(\cdot, 1) \longrightarrow 0,$$

uniformly with respect to (h_1, h_2) from bounded subsets of $L^2((0,T); H^{m-2} \times H^{m-1})$ and (w_0, θ_0) from bounded subsets of $\in H^m \times H^{m+1}(\mathbb{T}^2;\mathbb{R})$.

Proof. Given any fixed $\delta \in (0, \min\{1,T\})$, we denote by $(w_\delta, U_\delta, \Theta_\delta)$ the solution to (4.1) on $[0,\delta]$ issued from the initial state (w_0, θ_0), driven by $(h_1 + H_{1,\delta} + \nabla \wedge b_\delta, h_2 + H_{2,\delta})$, and having the velocity average $\aleph_\delta$. That is,

$$(W_\delta, \Theta_\delta) = S_\delta(w_0, \theta_0, h_1 + H_{1,\delta} + \nabla \wedge b_\delta, h_2 + H_{2,\delta}, \aleph_\delta)$$

and $U_\delta(x,t) = \Upsilon(W_\delta, \aleph_\delta)$. Then, in the limit $\delta \longrightarrow 0$, we study for the functions W_δ, U_δ, and Θ_δ an ansatz of the form

$$W_\delta = \overline{w}_\delta + z_\delta + r_\delta, \quad U_\delta = \overline{U}_\delta + Z_\delta + \Upsilon(r_\delta), \quad \Theta_\delta = \theta_\delta + s_\delta,$$

where r_δ and s_δ denote remainders and

$$z_\delta := v_\delta(\cdot, \delta^{-1}\cdot), \quad \overline{w}_\delta := \nabla \wedge \overline{U}_\delta, \quad \theta_\delta := \delta^{-1}\vartheta_\delta(\cdot, \delta^{-1}\cdot),$$
$$Z_\delta := \Upsilon(z_\delta, B_\delta(\delta^{-1}\cdot)).$$

Since $\overline{w}_\delta(\cdot, \delta) = 0$ in $\mathbb{T}^2$, the proof will be complete after the following convergence is obtained:

$$\|r_\delta(\cdot, \delta)\|_{m-1} + \|s_\delta(\cdot, \delta)\|_m \longrightarrow 0 \ \text{ as } \ \delta \longrightarrow 0. \tag{4.6}$$

Plugging this ansatz to the equations in (4.1), while recalling (4.2)–(4.4), the remainder profiles are seen to satisfy in $\mathbb{T}^2 \times [0, \delta]$ the problem

$$\begin{aligned} \partial_t r_\delta - \nu\Delta r_\delta + (\overline{U}_\delta + Z_\delta + \Upsilon(r_\delta)) \cdot \nabla r_\delta + \Upsilon(r_\delta) \cdot \nabla(\overline{w}_\delta + z_\delta) &= \partial_1 s_\delta + \Xi_\delta, \\ \partial_t s_\delta - \tau\Delta s_\delta + (\overline{U}_\delta + Z_\delta + \Upsilon(r_\delta)) \cdot \nabla s_\delta + \Upsilon(r_\delta) \cdot \nabla\theta_\delta &= \Lambda_\delta, \end{aligned} \tag{4.7}$$

with zero initial states $r_\delta(\cdot, 0) = s_\delta(\cdot, 0)$, and where

$$\Xi_\delta := \nu\Delta z_\delta - (Z_\delta \cdot \nabla)z_\delta + h_1, \quad \Lambda_\delta := \tau\Delta\theta_\delta - (Z_\delta \cdot \nabla)\theta_\delta + h_2.$$

To underscore that all estimates are uniform with respect to initial states and prescribed forces from bounded sets, we note that $(v_\delta, \delta^{-1}\vartheta_\delta)$ remains for each $\delta \in (0, \min\{1, T\})$ in a fixed bounded subset of $C^0([0,1]; H^m \times H^{m+1})$ when (w_0, θ_0) vary in a fixed bounded subset of $H^m \times H^{m+1}$. Now, we multiply in (4.7) with $(-\Delta)^{m-1} r_\delta$ and $(-\Delta)^m s_\delta$, respectively, followed by applications of Poincaré's inequality and the elliptic estimate $\|\Upsilon(\phi)\|_k \lesssim \|\phi\|_{k-1}$ for $k \in \mathbb{N}$, where "$\lesssim$" means "$\leqslant C$" and for a generic constant $C > 0$ independent of δ. For example, due to (4.5) and because A and b are fixed, we have

$$\begin{aligned} \sup_{t\in[0,\delta]} \|Z_\delta(\cdot, t)\|_{m+1} &\lesssim \sup_{t\in[0,\delta]} \|z_\delta(\cdot, t)\|_m + |A| + \|b\|_{L^1(0,\delta);L^1(\mathbb{T}^2;\mathbb{R}^2)} + 1 \\ &\lesssim \sup_{t\in[0,\delta]} \|z_\delta(\cdot, t)\|_m + 1. \end{aligned}$$

As a result of these steps, and by temporarily using abbreviations of the type $f = f(\cdot, t)$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big(\|r_\delta\|_{m-1}^2 + \|s_\delta\|_m^2\big) + \nu\|r_\delta\|_m^2 + \tau\|s_\delta\|_{m+1}^2 \\ &\qquad \lesssim \|r_\delta\|_{m-1}^2\big(\|z_\delta\|_m + \|r_\delta\|_m\big) + \|\overline{U}_\delta\|_{m+1}\big(\|r_\delta\|_{m-1}^2 + \|s_\delta\|_m^2\big) \\ &\qquad\quad + \|r_\delta\|_{m-1}\big(\|\theta_\delta\|_{m+1}\|s_\delta\|_m + \|s_\delta\|_m\|s_\delta\|_{m+1}\big) \\ &\qquad\quad + (\|z_\delta\|_m + 1)\big(\|r_\delta\|_{m-1}^2 + \|s_\delta\|_m^2\big) + \|r_\delta\|_{m-1}\|s_\delta\|_m \\ &\qquad\quad + \|\Xi_\delta\|_{m-2}\|r_\delta\|_m + \|\Lambda_\delta\|_{m-1}\|s_\delta\|_{m+1} \end{aligned} \tag{4.8}$$

for $t \in [0,\delta]$. Thanks to the continuous Sobolev embeddings $H^2(\mathbb{T}^2;\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{T}^2;\mathbb{R})$ and $H^1(\mathbb{T}^2;\mathbb{R}^N) \hookrightarrow L^4(\mathbb{T}^2;\mathbb{R}^N)$, $N \in \{1,2\}$, it follows that

$$
\begin{gathered}
\|r_\delta\|_{m-1}^2 \|r_\delta\|_m \leqslant \alpha \|r_\delta\|_m^2 + \alpha^{-1} C \|r_\delta\|_{m-1}^4, \\
\|r_\delta\|_{m-1} \|\theta_\delta\|_{m+1} \|s_\delta\|_m \leqslant \|\theta_\delta\|_{m+1} \big(\|r_\delta\|_{m-1}^2 + \|s_\delta\|_m^2\big), \\
\|r_\delta\|_{m-1} \|s_\delta\|_m \|s_\delta\|_{m+1} \leqslant \alpha \|s_\delta\|_{m+1}^2 + \alpha^{-1} C \big(\|r_\delta\|_{m-1}^4 + \|s_\delta\|_m^4\big) \\
\|r_\delta\|_{m-1} \|s_\delta\|_m \leqslant 2^{-1} \|r_\delta\|_{m-1}^2 + 2^{-1} \|s_\delta\|_m^2, \\
\|\Xi_\delta\|_{m-2} \leqslant \|z_\delta\|_m + \|z_\delta\|_m^2 + \|h_1\|_{m-2} + 1, \\
\|\Lambda_\delta\|_{m-1} \leqslant \|\theta_\delta\|_{m+1} + \|z_\delta\|_m^2 + \|\theta_\delta\|_{m+1}^2 + \|h_2\|_{m-1},
\end{gathered}
$$

where $\alpha > 0$ is small. The idea is then to observe that integrals involving several of the terms in (4.8) vanish when taking the limit $\delta \longrightarrow 0$. Indeed,

$$
\int_0^t \|f(\cdot,\sigma)\|_l \, d\sigma \leqslant \min\left\{\delta \int_0^1 \|f(\cdot,\delta\sigma)\|_l \, d\sigma,\ \int_0^\delta \|f(\cdot,\sigma)\|_l \, d\sigma\right\}
$$

for any $f \in L^1((0,\delta); H^l(\mathbb{T}^2;\mathbb{R}))$ with $l \geqslant 0$ and $t \in [0,\delta]$. For instance, recalling that $\vartheta_\delta(\cdot,0) = \delta\theta_0$ together with the choice of $(\zeta_\delta)_{\delta\in(0,\min\{1,T\})}$ in (4.4) yield $\sup_{t\in[0,\delta]} \|\theta_\delta(\cdot,t)\|_{m+1} = \mathcal{O}(1)$ as $\delta \longrightarrow 0$, we have

$$
\begin{gathered}
\lim_{\delta\to 0} \int^\delta (\|\Xi_\delta(\cdot,\sigma)\|_{m-2} + \|\Lambda_\delta(\cdot,\sigma)\|_{m-1}) \, d\sigma = 0, \\
\lim_{\delta\to 0} \int_0^\delta \|\overline{U}_\delta(\cdot,\sigma) + Z_\delta(\cdot,\sigma)\|_{m+1} \, d\sigma \leqslant \sup_{s\in[0,1]} \|\overline{U}(\cdot,s)\|_{m+1}, \\
\int_0^\delta \|\theta_\delta(\cdot,\sigma)\|_{m+1}^2 \, d\sigma \leqslant \delta^{-1} \sup_{s\in[0,1]} \|\vartheta_\delta(\cdot,s)\|_{m+1}^2 = \mathcal{O}(\delta) \text{ as } \delta \longrightarrow 0.
\end{gathered}
$$

Therefore, an application of Grönwall's lemma implies the existence of a constants $(C_\delta)_{\delta\in(0,\min\{1,T\})}$ satisfying $\lim_{\delta\to 0} C_\delta = 0$ and

$$
\|r_\delta(\cdot,t)\|_{m-1}^2 + \|s_\delta(\cdot,t)\|_m^2 \leqslant C_\delta + C \int_0^t \left(\|r_\delta(\cdot,\sigma)\|_{m-1}^4 + \|s_\delta(\cdot,\sigma)\|_m^4\right) d\sigma.
$$

Thus, the function

$$
\Psi_\delta(t) \coloneqq C_\delta + C \int_0^t \left(\|r_\delta(\cdot,\sigma)\|_{m-1}^4 + \|s_\delta(\cdot,\sigma)\|_m^4\right) d\sigma
$$

satisfies $d\Psi_\delta/dt \leqslant C\Psi_\delta^2$ and (4.6) follows by integrating this inequality. □

Remark 4.2. To conclude Theorem 1.1 and Corollary 1.2, we will apply Theorem 4.1, and Corollary 4.3 below, only with $b = 0$, $A = 0$, $\tau = 0$, and $\fint_{\mathbb{T}^2} \theta_0(x)\, dx = \fint_{\mathbb{T}^2} \theta_1(x)\, dx = 0$. Regarding the relevance of other choices for these parameters, see Remark 4.7.

Let $\widetilde{\mathscr{F}}_t$ be obtained by adding to $\mathscr{F}_t$ the space spanned by the smooth cutoff μ from Section 3.1.

Corollary 4.3. *Let $T > 0$, $m \geqslant 2$, $(w_0, \theta_0, \theta_1) \in H^m \times H^{m+1}(\mathbb{T}^2;\mathbb{R}) \times H^{m+1}(\mathbb{T}^2;\mathbb{R})$, $b \in C^\infty(\mathbb{T}^2 \times [0,1];\mathbb{R}^2)$, $A \in \mathbb{R}^2$, $(h_1, h_2) \in L^2((0,T); H^{m-2} \times H^{m-1})$, and $\tau \in C_0^\infty((0,1);\mathbb{R})$ with*

$$\int_0^1 \tau(s)\, ds = \int_{\mathbb{T}^2} (\theta_1 - \theta_0)(x)\, dx.$$

There exist $(\zeta_\delta)_{\delta \in (0,\min\{1,T\})} \subset L^2((0,1);\widetilde{\mathscr{F}}_t)$, with $\int_{\mathbb{T}^2} \zeta_\delta(x,t)\, dx = \delta\tau(t)$ for $\delta \in (0, \min\{1,T\})$ and almost all $t \in [0,1]$, so that in $H^{m-1} \times H^m(\mathbb{T}^2;\mathbb{R}^2)$ one has the convergence

$$\lim_{\delta \to 0} S_\delta\Big(w_0, \theta_0, h_1 + H_{1,\delta} + \nabla \wedge b_\delta, \\ h_2 + \delta^{-2}\zeta_\delta(\cdot, \delta^{-1}\cdot), \aleph_\delta\Big)\Big|_{t=\delta} = (v^b(\cdot,1), \theta_1), \tag{4.9}$$

where

$$b_\delta(\cdot,t) = \delta^{-1} b(\cdot, \delta^{-1}t), \quad \aleph_\delta(t) \coloneqq A_\delta(t) + \widetilde{B}(\delta^{-1}t)$$

for $t \in [0,\delta]$ and

$$\widetilde{B}(t) \coloneqq A + \int_0^t \int_{\mathbb{T}^2} b(x,s)\, dxds + e_2 \left(\int_0^t \int_0^s \tau(r)\, drds + t \int_{\mathbb{T}^2} \theta_0(x)\, dx \right)$$

for $t \in [0,1]$, while the function v^b solves

$$\begin{gathered} \partial_t v^b + (\overline{U} \cdot \nabla) v^b + (\Upsilon(v^b, \widetilde{B}) \cdot \nabla) \nabla \wedge \overline{U} = \nabla \wedge b, \\ v^b(\cdot, 0) = w_0. \end{gathered} \tag{4.10}$$

The limit in (4.9) is uniform with respect to (h_1, h_2) from bounded subsets of $L^2((0,T); H^{m-2} \times H^{m-1})$ and (w_0, θ_0) from bounded subsets of $\in H^m \times H^{m+1}(\mathbb{T}^2;\mathbb{R})$. Furthermore, if $(\theta_0, \theta_1) \in H^{m+1} \times H^{m+1}$ and $\tau = 0$, then one can use controls $(\zeta_\delta)_{\delta \in (0,\min\{1,T\})} \subset L^2((0,1);\mathscr{F}_t)$.

Proof. Let $\varepsilon > 0$. If $\tau = 0$, we apply Theorem 3.13 for each $\rho \in (0, \min\{1, T\})$ with initial state $\rho\theta_0$ and target state $\rho\theta_1$. In view of Remark 3.15, this provides controls $(\zeta_\rho)_{\rho\in(0,\min\{1,T\})} \in L^2((0,1);\mathscr{F}_t)$ such that the solution ϑ_ρ to the transport equation

$$\partial_t\vartheta_\rho + (\overline{U}\cdot\nabla)\vartheta_\rho = \zeta_\rho$$

with initial condition $\vartheta_\rho(\cdot,0) = \rho\theta_0$ satisfies

$$\|\vartheta_\rho(\cdot,1) - \rho\theta_1\|_m < \varepsilon\rho. \tag{4.11}$$

If some values of τ are nonzero, we use the version of Theorem 3.13 described in Remark 3.14, which leads instead to controls $(\zeta_\rho)_{\rho\in(0,\min\{1,T\})} \in L^2((0,1);\widetilde{\mathscr{F}}_t)$. According to Remark 3.15, the family $(\zeta_\rho)_{\rho\in(0,\min\{1,T\})}$ can be fixed such that

$$\|\zeta_\rho\|_{L^2((0,1);H^m)} = \mathscr{O}(\rho) \text{ as } \rho \longrightarrow 0.$$

Now, we define for $\rho \in (0, \min\{1, T\})$ the function v_ρ as the solution to

$$\partial_t v_\rho + \overline{U}\cdot\nabla v_\rho + \left(\Upsilon(v_\rho, \widetilde{B})\cdot\nabla\right)(\nabla\wedge\overline{U}) = \partial_1\vartheta_\rho + \nabla\wedge b$$

with initial condition $v_\rho(\cdot,0) = w_0$. Basic estimates and Remark 3.15 imply

$$\sup_{t\in[0,1]} \|\vartheta_\rho(\cdot,t)\|_{m+1} + \|v_\rho(\cdot,1) - v^b(\cdot,1)\|_m = \mathscr{O}(\rho) \text{ as } \rho \longrightarrow 0.$$

Thus, by combining Theorem 4.1 and (4.11), there exists $\delta_0 = \delta_0(\varepsilon) > 0$ such that one has for all $\delta \in (0, \delta_0)$ the estimate

$$\|(w_\delta, \theta_\delta)(\cdot,\delta) - (v^b(\cdot,1), \theta_1)\|_{H^{m-1}\times H^m(\mathbb{T}^2;\mathbb{R}^2)} < \varepsilon,$$

where

$$(w_\delta, \theta_\delta) \coloneqq S_\delta\left(w_0, \theta_0, h_1 + H_{1,\delta} + \nabla\wedge b_\delta, h_2 + \delta^{-2}\zeta_\delta(\cdot, \delta^{-1}\cdot), \aleph_\delta\right).$$

□

Theorem 1.1 follows now from the choice of controls $(\zeta_\delta)_{\delta\in(0,\min\{1,T\})} \subset L^2((0,1);\mathscr{F}_t)$ made via Corollary 4.3 for $w_0 = \nabla\wedge u_0$, $h_1 = \nabla\wedge f$, $h_2 = g$, $b = 0$, $A = 0$, $\tau = 0$, and $⨍_{\mathbb{T}^2}\theta_0(x)\,dx = ⨍_{\mathbb{T}^2}\theta_1(x)\,dx = 0$. Indeed, Lemma 3.16 implies in this case that $v^b(\cdot,1) = w_0$ for the solution v^b to (4.10). Moreover, the solution to (1.1) driven by the controls

$$\xi(\cdot,t) = \delta^{-2}\overline{H}(\cdot,\delta^{-1}t) - \delta^{-1}\Delta\overline{U}(\cdot,\delta^{-1}t),$$
$$\eta(\cdot,t) = \delta^{-2}\zeta_\delta(\cdot,\delta^{-1}t)$$

is for $t \in [0,\delta]$ given by $(u_\delta, \theta_\delta)$, where $u_\delta = \Upsilon(w_\delta, A_\delta)$ is the velocity associated with $(w_\delta, \theta_\delta) = S_\delta(w_0, \theta_0, H_{1,\delta}, h_2 + \delta^{-2}\zeta_\delta(\cdot, \delta^{-1}\cdot), A_\delta)$. Since $A_\delta(\delta) = 0$, the approximate controllability of u_δ follows from Corollary 4.3 by using the elliptic estimate

$$\|u_\delta(\cdot,\delta) - u_0\|_m \lesssim \|w_\delta(\cdot,\delta) - w_0\|_{m-1}.$$

The space $\mathscr{F}_t$ in Theorem 1.1 can be taken as the $(D_\omega + 8M)$-dimensional one from Theorem 3.13, recalling that D_ω is fixed in terms of the universal choice of $\overline{y}$ via Theorem 3.4. Regarding the space $\mathscr{F}_v$, from Theorem 3.4 and the representation $\overline{U}(x,t) = \sum_{i=1}^{4M+D_\omega} u_i(t)U_i(x)$ in (3.25), we know that $t \mapsto \overline{U}(\cdot,t)$ and $t \mapsto \Delta\overline{U}(\cdot,t)$ can be chosen as curves in possibly different but at most $(4M + D_\omega)$-dimensional spaces. Moreover, for q satisfying $\partial_t\overline{U} + (\overline{U}\cdot\nabla)\overline{U} = \nabla q$ in a neighborhood of $\mathbb{T}^2 \setminus \omega$, we can take $\overline{P} = -(1-\chi)q$, with χ from Section 3.1, and fix $\overline{H} = \partial_t\overline{U} + (\overline{U}\cdot\nabla)\overline{U} + \nabla\overline{P}$. Therefore, the dimension of $\mathscr{F}_v$ can be kept below or equal to

$$D_v := 12M + 3D_\omega + \frac{(4M + D_\omega)^2 - 4M - D_\omega}{2},$$

where the contribution

$$4M + D_\omega + \frac{(4M + D_\omega)^2 - 4M - D_\omega}{2}$$

to the above sum arises from grouping the nonlinear term

$$\left(\sum_{i=1}^{4M+D_\omega} u_i(t)U_i(x)\cdot\nabla\right)\sum_{i=1}^{4M+D_\omega} u_i(t)U_i(x)$$

with respect to the common factors $u_i(t)u_i(t)$ and $u_j(t)u_l(t) = u_l(t)u_j(t)$ with $i,j,l \in \{1,\dots,4M+D_\omega\}$ and $l \neq j$. The pressure $\overline{P}$ does not add any dimensions as one may group profiles with respect to common control coefficients in the representation of $\overline{H}$. Summarized, plugging (3.25) into (4.2) yields

$$\overline{H}(x,t) + \Delta\overline{U}(x,t) = \sum_{i=1}^{D_v} w_i(t)W_i(x),$$

for coefficients $w_1,\dots,w_{D_v} \in L^2((0,1);\mathbb{R})$ and universal profiles $W_1,\dots,W_{D_v} \in C^\infty(\mathbb{T}^2;\mathbb{R}^2)$ supported in ω.

Remark 4.4. In this remark, we assume that $\mathbb{T}^2 \setminus \omega$ is simply-connected and that ω contains a closed square of length L. Then, instead of using the proof of

Theorem 3.4, one can define the profile $\overline{y}$ appearing in (3.18) as a function only of time (*c.f.* Remark 3.6); in that case, $D_{\omega} = 2$. If $2\pi/L$ is an integer, one may slightly increase L so that the resulting closed square of side-length L is still contained in ω. Subsequently, an open covering by $M = \lceil 2\pi/L \rceil^2$ overlapping squares of side-length L can be defined with the properties required in Section 3.1. See also Example 3.1 for a concrete definition of such a covering. Moreover, due to (3.18) and the constant-in-space choice of $\overline{y}$, the profile $\overline{H}$ from (4.2) can be taken as a curve in an explicitly constructed $(2 + 14M + 8M^2)$-dimensional space. Since $\Delta\overline{y} = 0$ for spatially constant $\overline{y}$, the function $\Delta\overline{U}(\cdot, t)$ belongs for each $t \in [0, 1]$ to a $4M$-dimensional space. As a result, we can choose $\mathscr{F}_v$ of dimension $2 + 18\lceil 2\pi/L \rceil^2 + 8\lceil 2\pi/L \rceil^4$ and $\mathscr{F}_t$ of dimension $2 + 8\lceil 2\pi/L \rceil^2$. See Section 1.4 for an explicit list of functions that span these spaces.

4.2 Proof of Corollary 1.2

This section briefly recalls a strategy from [40], rendering Corollary 1.2 as a consequence of Theorem 1.1. Hereto, let $m \in \mathbb{N}$, which corresponds to $k \geqslant 2$ in Corollary 1.2. If $k \in \{0, 1\}$, the statement follows from a density argument, as one can approximate the target states by smooth functions.

Step 1. Stability

Due to the well-posedness of (4.1) in the considered spaces, there exists a small time $\sigma > 0$ such that for each $\delta \in [0, \sigma]$ and $(a, b) \in H^m \times H^{m+1}$ with $\|(a, b) - (w_0, \theta_1)\|_{H^m \times H^{m+1}} < \varepsilon/2$, one has

$$\|S_{\delta}(a, b, \nabla \wedge f(T - \delta + \cdot), g(T - \delta + \cdot), 0)|_{t=\delta} - (w_0, \theta_1)\|_{H^m \times H^{m+1}} < \varepsilon.$$

Step 2. Regularization

On a time interval $[0, T - \delta_0]$, where $\delta_0 \in (0, \sigma]$ will be selected below and σ is the number from the previous step, all controls are set to zero. Then, the smoothing effects of the considered viscous and thermally diffusive Boussinesq system imply that the solution (u, θ) to (1.1) in the Leray-Hopf class, with initial data (u_0, θ_0) and forces (f, g), belongs to the space

$$C^0((0, T - \delta_0]; H^{m+1}(\mathbb{T}^2; \mathbb{R}^2) \times H^{m+1}) \cap L^2((0, T - \delta_0); H^{m+2}(\mathbb{T}^2; \mathbb{R}^2) \times H^{m+2}).$$

As a result, one has

$$(u, \theta)(\cdot, t) \in V^{m+2} \times H^{m+2}$$

for almost all $t \in (0, T - \delta_0)$, which implies

$$(\widetilde{w}_0, \widetilde{\theta}_0) \coloneqq (\nabla \wedge u, \theta)(\cdot, T - \delta_0) \in H^{m+1} \times H^{m+2}$$

for a well-chosen $\delta_0 \in (0, \sigma]$, which is fixed from now on. We refer also to [6, 53–55] regarding Leray-Hopf type solutions and smoothing effects, presented for the more challenging case of domains with boundaries. Moreover, since the goal is to prove approximate controllability, we can without loss of generality assume that the targets u_1 and θ_1 are smooth.

Step 3. Control strategy

As the following mechanism is known from [40], we simplify here the presentation by assuming $f = g = 0$ in (1.1). Moreover, for the resolving operator S_T of (4.1) with time $T > 0$, we abbreviate $S_T(\cdot, \cdot) = S_T(\cdot, \cdot, 0, 0, 0)$. The next theorem states two scaling limits from [40]. Similar results have been obtained in [5] for the primitive equations.

Theorem 4.5. [40, Theorem 3.4] *Let $k \in \mathbb{N}$, T, zero average $q \in C^\infty(\mathbb{T}^2; \mathbb{R})$, and $(w_0, \theta_0) \in H^{k+1} \times H^{k+2}$. Further, denote by $\Pi_1(w, \theta) \coloneqq w$ the projection to the vorticity component of a solution (w, θ) to (4.1). Then, as $\delta \longrightarrow 0$, the limits*

$$\Pi_1 S_\delta(w_0, \theta_0 - \delta^{-1} q)|_{t=\delta} \longrightarrow w_0 - \partial_1 q, \tag{4.12}$$

$$S_\delta(w_0 + \delta^{-1/2} q, \theta_0)|_{t=\delta} - (\delta^{-1/2} q, 0) \longrightarrow (w_0 - (\Upsilon(q) \cdot \nabla) q, \theta_0) \tag{4.13}$$

hold in H^k and $H^k \times H^{k+1}$, respectively.

Remark 4.6. Lets us briefly describe the idea employed in [40] for proving Theorem 4.5. Regarding (4.12), the following ansatz is made as $\delta \longrightarrow 0$:

$$S_\delta(w_0, \theta_0 - \delta^{-1} q)(x, t) + (0, \delta^{-1} q(x)) = \begin{bmatrix} w_0(x) \\ \theta_0(x) \end{bmatrix} - \begin{bmatrix} \delta^{-1} t \partial_1 q(x) \\ \delta^{-1} t \tau \Delta q(x) + \delta^{-1} t (Q_\delta(x, t) \cdot \nabla) q(x) \end{bmatrix} + R_\delta(x, t),$$

where $R_\delta(x, t)$ denotes a supposedly small remainder term and

$$Q_\delta(x, t) \coloneqq \Upsilon \left(w_0 - \frac{\delta^{-1} t \partial_1 q}{2} \right)(x)$$

for $(x, t) \in \mathbb{T}^2 \times [0, \delta]$. Via energy estimates for the equation satisfied by R_δ, it is seen that $R_\delta(\cdot, \delta)$ vanishes in $H^k \times H^{k+1}$ as $\delta \longrightarrow 0$. Concerning (4.13), the ansatz used in [40] is of the form

$$S_\delta(w_0 + \delta^{-1/2}q, \theta_0)(x,t) - (\delta^{-1/2}q(x), 0) =$$
$$\begin{bmatrix} w_0(x) \\ \theta_0(x) \end{bmatrix} - \begin{bmatrix} \delta^{-1}t\,(\Upsilon(q)(x)\cdot\nabla)\,q(x) - \delta^{-1/2}t\nu\Delta q(x) \\ 0 \end{bmatrix} + R_\delta(x,t)$$

for $(x,t) \in \mathbb{T}^2 \times [0,\delta]$. Also in this case, the remainder $R_\delta(\cdot,\delta)$ is seen to vanish in $H^k \times H^{k+1}$ as $\delta \longrightarrow 0$.

The convergence results of Theorem 4.5 will be combined with the fact that $\mathscr{E}$ from (1.4) contains $\pm\sin(x\cdot n)$ and $\pm\cos(x\cdot n)$ for all $n \in \mathbb{Z}^2 \setminus \{0\}$. Let us recall that

$$\mathscr{E} = \left\{ q_0 + (\Upsilon(q_1)\cdot\nabla)\,q_1 + (\Upsilon(q_2)\cdot\nabla)\,q_2 \mid q_0, q_1, q_2 \in \operatorname{span}_{\mathbb{R}} \mathscr{E}_0 \right\},$$

where $\mathscr{E}_0$ collects all functions $\sin(x\cdot n)$ and $\cos(x\cdot n)$ for $n \in \mathbb{N}\times\mathbb{N}_0$. In particular, there exists an integer $N \geqslant 0$ and

$$q_0, q_1, \ldots, q_{2N} \in \operatorname{span}_{\mathbb{R}} \mathscr{E}_0, \quad Q_0, Q_1, \ldots, Q_{2N} \in C^\infty(\mathbb{T}^2;\mathbb{R})$$

such that

$$\|W_1 - w_1\|_m < \frac{\varepsilon}{3}, \tag{4.14}$$

where

$$W_1 \coloneqq \widetilde{w}_0 - q_0 - \sum_{i=1}^{2N}(\Upsilon(q_i)\cdot\nabla)q_i$$

and

$$q_0 = \partial_1 Q_0, \quad q_1 = \partial_1 Q_1, \quad \ldots, \quad q_{2N} = \partial_1 Q_{2N}.$$

Starting a trajectory at time $T_0 = T - \delta_0$ from the state $(\widetilde{w}_0, \widetilde{\theta}_0)$, the following steps i)-iv) demonstrate how the vorticity in the Boussinesq system (4.1) can be steered faster than $(T - T_0)/(10N+1)$, and up to any small error $\overline{\varepsilon} > 0$ with respect to the H^m-norm, to the value

$$\widetilde{w}_0 - q_0 - (\Upsilon(q_1)\cdot\nabla)q_1.$$

Thanks to (4.14), by repeating the below argument $(2N-1)$-times and choosing $\overline{\varepsilon}$, one can build a piece-wise (in time) defined trajectory so that the associated vorticity reaches W_1 in H^m up to any prescribed error $\widetilde{\varepsilon}$. After a final application of Corollary 4.3 to steer the temperature in H^{m+1} as close to θ_1 as required, the proof is complete.

i) For any $\varepsilon_1 > 0$, we take $0 < \delta_3 < (T - T_0)/(10N + 1)$ so small that (4.13) implies

$$\|S_{\delta_3}\left(\widetilde{w}_0 - \delta_3^{-1/2} q_1, \widetilde{\Theta}\right)|_{t=\delta_3} - (\widetilde{w}_0 - \delta_3^{-1/2} q_1 - (\Upsilon(q_1) \cdot \nabla) q_1, \widetilde{\Theta})\|_{H^m \times H^{m+1}} < \varepsilon_1,$$

where $\widetilde{\Theta}$ denotes a fixed element of H^{m+1} (*e.g.*, choose $\widetilde{\Theta} = 0$).

ii) We apply (4.12) and Corollary 4.3, the latter with target temperature $\widetilde{\Theta}$, in order to fix a small $0 < \delta_2 < (T - T_0)/(10N + 1)$ of the form $\delta_2 = \delta_{2,1} + \delta_{2,2}$ and a control $\zeta^0 \in L^2((0, 1); \mathscr{F}_t)$ such that

$$\begin{aligned}\|S_{\delta_3}&\left(\widetilde{S}_{\delta_2}\left(\widetilde{w}_0, \widetilde{\theta}_0 - \delta_{2,1}^{-1}\delta_3^{-1/2} Q_1\right)|_{t=\delta_2}\right)|_{t=\delta_3} \\ &- (\widetilde{w}_0 - \delta_3^{-1/2} q_1 - (\Upsilon(q_1) \cdot \nabla) q_1, \widetilde{\Theta})\|_{H^m \times H^{m+1}} < \varepsilon_1,\end{aligned}$$

denoting

$$\begin{aligned}\widetilde{S}_{\delta_2}(A, B) \coloneqq S_{\delta_2}\big(&A, B, \mathbb{I}_{[\delta_{2,1},\delta_{2,2}]} H_{1,\delta_{2,2}}(\cdot, \cdot - \delta_{2,1}), \\ &\delta_{2,2}^{-2}\mathbb{I}_{[\delta_{2,1},\delta_{2,2}]}\zeta^0(\cdot, \delta_{2,2}^{-1}(\cdot - \delta_{2,1})), \mathbb{I}_{[\delta_{2,1},\delta_{2,2}]} A_{\delta_{2,2}}(\cdot - \delta_{2,1})\Big).\end{aligned}$$

iii) By an application of Corollary 4.3 with target temperature $\widetilde{\theta}_0 - \delta_{2,1}^{-1}\delta_3^{-1/2} Q_1$, we fix $\zeta^1 \in L^2((0, 1); \mathscr{F}_t)$ and $\delta_1 < (T - T_0)/(10N + 1)$ so small that

$$\begin{aligned}\|S_{\delta_3}&\left(\widetilde{S}_{\delta_2}\left(S_{\delta_1}\left(\widetilde{w}_0, \widetilde{\theta}_0, H_{1,\delta_1}, \delta_1^{-2}\zeta^1(\cdot, \delta_1^{-1}\cdot), A_{\delta_1}\right)|_{t=\delta_1}\right)|_{t=\delta_2}\right)|_{t=\delta_3} \\ &- (\widetilde{w}_0 - \delta_3^{-1/2} q_1 - (\Upsilon(q_1) \cdot \nabla) q_1, \widetilde{\Theta})\|_{H^m \times H^{m+1}} < \varepsilon_1.\end{aligned}$$

Now, for any given $\overline{\varepsilon} > 0$, we select the number $\varepsilon_1 > 0$ used in the steps above (hence, we fix a choice of $\delta_1 = \delta_1(\delta_2, \delta_3)$, $\delta_2 = \delta_2(\delta_3)$, and δ_3) and determine $0 < \delta_5 < (T - T_0)/(10N + 1)$ via (4.12) such that

$$\begin{aligned}\|\Pi_1 S_{\delta_5}&\left(\widetilde{w}_0 - \delta_3^{-1/2} q_1 - (\Upsilon(q_1) \cdot \nabla) q_1, \delta_5^{-1}(\delta_3^{-1/2} Q_1 - Q_0)\right)|_{t=\delta_5} \\ &- (\widetilde{w}_0 - q_0 - (\Upsilon(q_1) \cdot \nabla) q_1)\|_{H^m \times H^{m+1}} < \overline{\varepsilon}.\end{aligned}$$

iv) By applying again Corollary 4.3, we select $0 < \delta_4 < (T - T_0)/(10N + 1)$ and $\zeta^2 \in L^2((0, 1); \mathscr{F}_t)$ such that

$$\begin{aligned}\|\Pi_1 S_{\delta_5}&\left(S_{\delta_4}\left(S_3, H_{1,\delta_4}, +\delta_4^{-2}\zeta^2(\cdot, \delta_4^{-1}\cdot), A_{\delta_4}\right)|_{t=\delta_4}\right)|_{t=\delta_5} \\ &- (\widetilde{w}_0 - q_0 - (\Upsilon(q_1) \cdot \nabla) q_1, \widetilde{\Theta})\|_{H^m \times H^{m+1}} < \overline{\varepsilon},\end{aligned}$$

where

$$S_3 := S_{\delta_3}\left(\widetilde{S}_{\delta_2}\left(S_{\delta_1}\left(\widetilde{w}_0, \widetilde{\theta}_0, H_{1,\delta_1}, \delta_1^{-2}\zeta^1(\cdot, \delta_1^{-1}\cdot), A_{\delta_1}\right)|_{t=\delta_1}\right)|_{t=\delta_2}\right)|_{t=\delta_3}.$$

The so-obtained controls for the velocity and temperature are zero on the union of time intervals

$$\begin{gathered}\left[T_0+\delta_1, T_0+\delta_1+\delta_{2,1}\right], \\ \left[T_0+\delta_1+\delta_2, T_0+\sum_{l=1}^{3}\delta_l\right], \quad \left[T_0+\sum_{l=1}^{4}\delta_l, T_0+\sum_{l=1}^{5}\delta_l\right],\end{gathered}$$

while assuming possibly nonzero values in the respective spaces $\mathscr{F}_v$ and $\mathscr{F}_t$ on the union of the time intervals

$$[T_0, T_0+\delta_1], \quad [T_0+\delta_1+\delta_{2,1}, T_0+\delta_1+\delta_2], \quad \left[T_0+\sum_{l=1}^{3}\delta_l, T_0+\sum_{l=1}^{4}\delta_l\right].$$

Remark 4.7. The proof of Corollary 1.2 extends to initial- and target states with nonzero average. Hereto, one has to add two short average control stages at the beginning and at the end of "Step 3. Control strategy". To illustrate this, let us observe that the velocity and temperature averages of solutions to (1.1) (with body forces of zero average) behave formally like

$$\begin{gathered}\int_{\mathbb{T}^2} u(x,t)\,dx = \int_{\mathbb{T}^2} u_0(x)\,dx + \int_0^t\int_{\mathbb{T}^2}\theta(x,s)e_2\,dxds + \int_0^t\int_{\omega}\xi(x,s)\,dxds, \\ \int_{\mathbb{T}^2}\theta(x,t)\,dx = \int_{\mathbb{T}^2}\theta_0(x)\,dx + \int_0^t\int_{\omega}\eta(x,s)\,dxds\end{gathered}$$

for $t \in [0,T]$.

To begin with, suppose that u_0 and u_1 are of average $A_0 = (A_{0,1}, A_{0,2}) \in \mathbb{R}^2$ and $A_1 = (A_{1,1}, A_{1,2}) \in \mathbb{R}^2$, respectively. Further, assume that the averages of θ_0 and θ_1 are $\tau_0 \in \mathbb{R}$ and $\tau_1 \in \mathbb{R}$, respectively. Then, fix two vector fields $\mathscr{a}, \mathscr{b} \in C^\infty(\mathbb{T}^2;\mathbb{R}^2)$ with $\operatorname{supp}(\mathscr{a}) \cup \operatorname{supp}(\mathscr{b}) \subset \omega$ and

$$\int_{\mathbb{T}^2}\mathscr{a}(x)\,dx = (1,0), \quad \int_{\mathbb{T}^2}\mathscr{b}(x)\,dx = (0,1).$$

Moreover, choose profiles $\lambda_{0,1}, \lambda_{0,2}, \lambda_{1,1}, \lambda_{1,2}, r_0, r_1 \in C_0^\infty((0,1);\mathbb{R})$ such that

$$\int_0^1 \lambda_{0,1}(s)\, ds = -A_{0,1}, \quad \int_0^1 \lambda_{1,1}(s)\, ds = A_{1,1},$$
$$\int_0^1 r_0(s)\, ds = -\tau_0, \quad \int_0^1 \lambda_{0,2}(s)\, ds + \int_0^1 \int_0^t r_0(s)\, dsdt = -A_{0,2} - \tau_0,$$
$$\int_0^1 r_1(s)\, ds = \tau_1, \quad \int_0^1 \lambda_{1,2}(s)\, ds + \int_0^1 \int_0^t r_1(s)\, dsdt = A_{1,2}$$

Now, the above proof of Corollary 1.2 is adapted as described in the following points.

1) Since the arguments in "Step 1. Stability" and "Step 2. Regularization" likewise work for initial data with nonzero average, no significant changes are made there: the velocity and temperature equations with zero average body forces and zero controls preserve the averages of the initial data until the time $T - \delta_0$.

2) Before starting with "Step 3. Control strategy", we apply Corollary 4.3 with $b = \lambda_{0,1}\mathscr{a} + \lambda_{0,2}\mathscr{b}$, $A = A_0$, $\tau = r_0$, and zero target temperature. Notably, the velocity and temperature averages are steered exactly to zero by this preliminary application of Corollary 4.3; see also Remark 3.14.

3) The original target vorticity w_1 in "Step 3. Control strategy" is replaced by $w_{A_1} = \widetilde{v}_b(\cdot, 0)$, where $\widetilde{v}_b$ solves backwards in time the problem (4.10) with prescribed endpoint $v^b(\cdot, 1) = w_1$ and $b = \lambda_{1,1}\mathscr{a} + \lambda_{1,2}\mathscr{b}$.

4) At the end of "Step 3", we insert another application Corollary 4.3, now with $b = \lambda_{1,1}\mathscr{a} + \lambda_{1,2}\mathscr{b}$, $A = 0$, $\tau = r_1$, zero initial temperature average, and desired target temperature.

References

[1] A. A. Agrachev, *Some open problems*, Geometric control theory and sub-Riemannian geometry, 2014, pp. 1–13.

[2] A. A. Agrachev and A. V. Sarychev, *Navier-Stokes equations: controllability by means of low modes forcing*, J. Math. Fluid Mech. **7** (2005), no. 1, 108–152.

[3] A. A. Agrachev and A. V. Sarychev, *Controllability of 2D Euler and Navier-Stokes equations by degenerate forcing*, Comm. Math. Phys. **265** (2006), no. 3, 673–697.

[4] A. A. Agrachev and A. V. Sarychev, *Solid controllability in fluid dynamics*, Instability in models connected with fluid flows. I, 2008, pp. 1–35.

[5] P.-M. Boulvard, P. Gao, and V. Nersesyan, *Controllability and ergodicity of three dimensional primitive equations driven by a finite-dimensional force*, Arch. Ration. Mech. Anal. **247** (2023), no. 1, Paper No. 2, 49.

[6] F. W. Chaves-Silva, E. Fernández-Cara, K. Le Balc'h, J. L. F. Machado, and D. A. Souza, *Global controllability of the Boussinesq system with Navier-slip-with-friction and Robin boundary conditions*, SIAM J. Control Optim. **61** (2023), no. 2, 484–510.

[7] P. Constantin and C. R. Doering, *Infinite Prandtl number convection*, J. Statist. Phys. **94** (1999), no. 1-2, 159–172.

[8] J.-M. Coron, *On the controllability of the 2-D incompressible Navier-Stokes equations with the Navier slip boundary conditions*, ESAIM Contrôle Optim. Calc. Var. **1** (1995/96), 35–75.

[9] J.-M. Coron, *On the controllability of 2-D incompressible perfect fluids*, J. Math. Pures Appl. **75** (1996), no. 2, 155–188.

[10] J.-M. Coron, *Control and nonlinearity*, Mathematical Surveys and Monographs, vol. 136, American Mathematical Society, Providence, RI, 2007.

[11] J.-M. Coron and A. V. Fursikov, *Global exact controllability of the 2D Navier-Stokes equations on a manifold without boundary*, Russian J. Math. Phys. **4** (1996), no. 4, 429–448.

[12] J.-M. Coron and P. Lissy, *Local null controllability of the three-dimensional Navier-Stokes system with a distributed control having two vanishing components*, Invent. Math. **198** (2014), no. 3, 833–880.

[13] J.-M. Coron, F. Marbach, and F. Sueur, *Small-time global exact controllability of the Navier-Stokes equation with Navier slip-with-friction boundary conditions*, J. Eur. Math. Soc. (JEMS) **22** (2020), no. 5, 1625–1673.

[14] J.-M. Coron, F. Marbach, F. Sueur, and P. Zhang, *Controllability of the Navier-Stokes equation in a rectangle with a little help of a distributed phantom force*, Ann. PDE **5** (2019), no. 2, Art. 17.

[15] J.-M. Coron and S. Xiang, *Small-time global stabilization of the viscous Burgers equation with three scalar controls*, J. Math. Pures Appl. (9) **151** (2021), 212–256.

[16] J.-M. Coron, S. Xiang, and P. Zhang, *On the global approximate controllability in small time of semiclassical 1-D Schrödinger equations between two states with positive quantum densities*, J. Differential Equations **345** (2023), 1–44.

[17] A. Duca and V. Nersesyan, *Bilinear control and growth of Sobolev norms for the nonlinear Schrödinger equation*, J. Eur. Math. Soc. (JEMS) **27** (2025), no. 6, 2603–2622.

[18] S. Ervedoza, K. Le Balc'h, and M. Tucsnak, *Reachability results for perturbed heat equations*, J. Funct. Anal. **283** (2022), no. 10, Paper No. 109666, 61.

[19] E. Fernández-Cara, S. Guerrero, O. Yu. Imanuvilov, and J.-P. Puel, *Local exact controllability of the Navier-Stokes system*, J. Math. Pures Appl. (9) **83** (2004), no. 12, 1501–1542.

[20] E. Fernández-Cara, S. Guerrero, O. Yu. Imanuvilov, and J.-P. Puel, *Some controllability results for the N-dimensional Navier-Stokes and Boussinesq systems with $N-1$ scalar controls*, SIAM J. Control Optim. **45** (2006), no. 1, 146–173.

[21] C. Foias, O. Manley, and R. Temam, *Attractors for the Bénard problem: Existence and physical bounds on their fractal dimension*, Nonlinear Anal., Theory Methods Appl. **11** (1987), 939–967 (English).

[22] J. Földes and D. P. Herzog, *Small-time asymptotics for hypoelliptic diffusions*, arXiv preprint (2024), available at https://doi.org/10.48550/arXiv.2412.11323.

[23] A. V. Fursikov and O. Yu. Imanuvilov, *Local exact boundary controllability of the Boussinesq equation*, SIAM J. Control Optim. **36** (1998), no. 2, 391–421.

[24] A. V. Fursikov and O. Yu. Imanuvilov, *Exact controllability of the Navier-Stokes and Boussinesq equations*, Uspekhi Mat. Nauk **54** (1999), no. 3(327), 93–146.

[25] O. Glass, *Exact boundary controllability of 3-D Euler equation*, ESAIM Control Optim. Calc. Var. **5** (2000), 1–44.

[26] S. Guerrero, *Local exact controllability to the trajectories of the Boussinesq system*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **23** (2006), no. 1, 29–61.

[27] M. Hairer and J. C. Mattingly, *Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing*, Ann. of Math. (2) **164** (2006), no. 3, 993–1032.

[28] I. Kukavica, M. Novack, and V. Vicol, *Exact boundary controllability for the ideal magneto-hydrodynamic equations*, J. Differential Equations **318** (2022), 94–112.

[29] S. Kuksin, V. Nersesyan, and A. Shirikyan, *Exponential mixing for a class of dissipative PDEs with bounded degenerate noise*, Geom. Funct. Anal. **30** (2020), no. 1, 126–187.

[30] J. Liao, *Global exact controllability of the viscous and resistive MHD system in a rectangle thanks to the lateral sides and to distributed phantom forces*, ESAIM Control Optim. Calc. Var. **30** (2024), Paper No. 67, 40.

[31] J. Liao, F. Sueur, and P. Zhang, *Global controllability of the Navier-Stokes equations in the presence of curved boundary with no-slip conditions*, J. Math. Fluid Mech. **24** (2022), no. 3, Paper No. 71, 32.

[32] J. Liao, F. Sueur, and P. Zhang, *Smooth controllability of the Navier-Stokes equation with Navier conditions: application to Lagrangian controllability*, Arch. Ration. Mech. Anal. **243** (2022), no. 2, 869–941.

[33] J.-L. Lions, *Exact controllability for distributed systems. Some trends and some problems*, Applied and industrial mathematics (Venice, 1989), 1991, pp. 59–84.

[34] J.-L. Lions and E. Zuazua, *Exact boundary controllability of Galerkin's approximations of Navier-Stokes equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **26** (1998), no. 4, 605–621.

[35] A. Majda, *Introduction to PDEs and waves for the atmosphere and ocean*, Courant Lecture Notes in Mathematics, vol. 9, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[36] J. C. Mattingly and É. Pardoux, *Malliavin calculus for the stochastic 2D Navier-Stokes equation*, Comm. Pure Appl. Math. **59** (2006), no. 12, 1742–1790.

[37] V. Nersesyan, *Approximate controllability of Lagrangian trajectories of the 3D Navier-Stokes system by a finite-dimensional force*, Nonlinearity **28** (2015), no. 3, 825–848.

[38] V. Nersesyan, *A proof of approximate controllability of the 3D Navier-Stokes system via a linear test*, SIAM J. Control Optim. **59** (2021), no. 4, 2411–2427.

[39] V. Nersesyan, *The complex Ginzburg-Landau equation perturbed by a force localised both in physical and Fourier spaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **25** (2024), no. 2, 1203–1223.

[40] V. Nersesyan and M. Rissel, *Global controllability of Boussinesq flows by using only a temperature control*, arXiv preprint (2024), available at https://doi.org/10.48550/arXiv:2404.09903.

[41] V. Nersesyan and M. Rissel, *Localized and degenerate controls for the incompressible Navier-Stokes system*, Comm. Pure Appl. Math. **78** (2025), no. 7, 1285–1319.

[42] H. Nersisyan, *Controllability of the 3D compressible Euler system*, Comm. Partial Differential Equations **36** (2011), no. 9, 1544–1564.

[43] D. Phan and S. S. Rodrigues, *Approximate controllability for Navier-Stokes equations in 3D rectangles under Lions boundary conditions*, J. Dyn. Control Syst. **25** (2019), no. 3, 351–376.

[44] M. Rissel, *Global controllability of Boussinesq channel flows only through the temperature*, arXiv preprint (2024), available at https://doi.org/10.48550/arXiv.2411.02200.

[45] M. Rissel and Y.-G. Wang, *Small-time global approximate controllability for incompressible MHD with coupled Navier slip boundary conditions*, J. Math. Pures Appl. (9) **190** (2024), Paper No. 103601, 66.

[46] S. S. Rodrigues, *Navier-Stokes equation on the rectangle controllability by means of low mode forcing*, J. Dyn. Control Syst. **12** (2006), no. 4, 517–562.

[47] S. S. Rodrigues, *Navier-Stokes equation on a plane bounded domain: continuity properties for controllability*, Taming heterogeneity and complexity of embedded control, 2007, pp. 585–615.

[48] A. Sarychev, *Controllability of the cubic Schroedinger equation via a low-dimensional source term*, Math. Control Relat. Fields **2** (2012), no. 3, 247–270.

[49] A. Shirikyan, *Approximate controllability of three-dimensional Navier-Stokes equations*, Comm. Math. Phys. **266** (2006), no. 1, 123–151.

[50] A. Shirikyan, *Exact controllability in projections for three-dimensional Navier–Stokes equations*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **24** (2007), no. 4, 521–537.

[51] A. Shirikyan, *Euler equations are not exactly controllable by a finite-dimensional external force*, Phys. D **237** (2008), no. 10-12, 1317–1323.

[52] A. Shirikyan, *Control theory for the Burgers equation: Agrachev-Sarychev approach*, Pure Appl. Funct. Anal. **3** (2018), no. 1, 219–240.

[53] R. Temam, *Behaviour at time $t = 0$ of the solutions of semilinear evolution equations*, J. Differential Equations **43** (1982), no. 1, 73–92.

[54] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics.*, 2nd ed., Appl. Math. Sci., vol. 68, New York, NY: Springer, 1997 (English).

[55] R. Temam, *Navier-Stokes equations*, AMS Chelsea Publishing, Providence, RI, 2001. Theory and numerical analysis, Reprint of the 1984 edition.

[56] S. Xiang, *Small-time local stabilization of the two-dimensional incompressible Navier-Stokes equations*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **40** (2023), no. 6, 1487–1511.