Localized and degenerate controls for the incompressible Navier–Stokes system

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Abstract

We consider the global approximate controllability of the two-dimensional incompressible Navier–Stokes system driven by a physically localized and degenerate force. In other words, the fluid is regulated via four scalar controls that depend only on time and appear as coefficients in an effectively constructed driving force supported in a given subdomain. Our idea consists of squeezing low mode controls into a small region, essentially by tracking their actions along the characteristic curves of a linearized vorticity equation. In this way, through explicit constructions and by connecting Coron's return method with recent concepts from geometric control, the original problem for the nonlinear Navier–Stokes system is reduced to one for a linear transport equation steered by a global force. This article can be viewed as an attempt to tackle a well-known open problem due to Agrachev.

Keywords: approximate controllability, degenerate controls, localized controls, Navier–Stokes equations, return method, transported Fourier modes
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1 Introduction

The study of fluid mechanics has ever since been intertwined with aspirations to not only observe, but also to manipulate or regulate flows of liquids. Modern control theory, to this end, offers suitable notions that allow formalizing such endeavors. The question of controllability, in particular, is to ask whether and in which sense external influences (the controls) can cause a system to transition between designated states. From a mathematical stance, and due to the nonlinear nature of the involved constituents, several types of difficulties may arise. These are often related to the size of the prescribed data, how the controls are allowed to enter the system, and the overall degree of degeneracy in the formulated controllability problem. In this sense, the present article sets out to accommodate three ambitious requirements. First, the admissible states might be located far away from each other. Second, the fluid should be acted upon merely in a subdomain of small area. Third, it is desired that the driving force can be expressed through explicit formulas in terms of a fixed number of unknown control parameters.

We establish the global approximate controllability for incompressible viscous fluids on the torus $\mathbb{T}^2 := \mathbb{R}^2/2\pi\mathbb{Z}^2$ by means of a degenerate force which is physically localized in a given subdomain $\Omega \subset \mathbb{T}^2$. The keyword "global" refers to the admissible distance between initial and target profiles being unlimited, the property "degenerate" emphasizes that there are only few degrees of freedom, while "localized" means here that Ω can be any subdomain containing two curves $C_1, C_2 \subset \Omega$ rendering the cut torus $\mathbb{T}^2 \setminus (C_1 \cup C_2)$ simply-connected (*cf.* Figure 1). More precisely, the force employed as a control is supported in the control region Ω and explicitly depends, through a formula, on four unknown control parameters. Aside from these four parameters, which are functions only of time, and a scaling constant, the external force shall be fixed independently of the prescribed data. Our efforts contrast the common approach of searching localized interior controls in the form of purely abstract elements of infinite-dimensional function spaces.

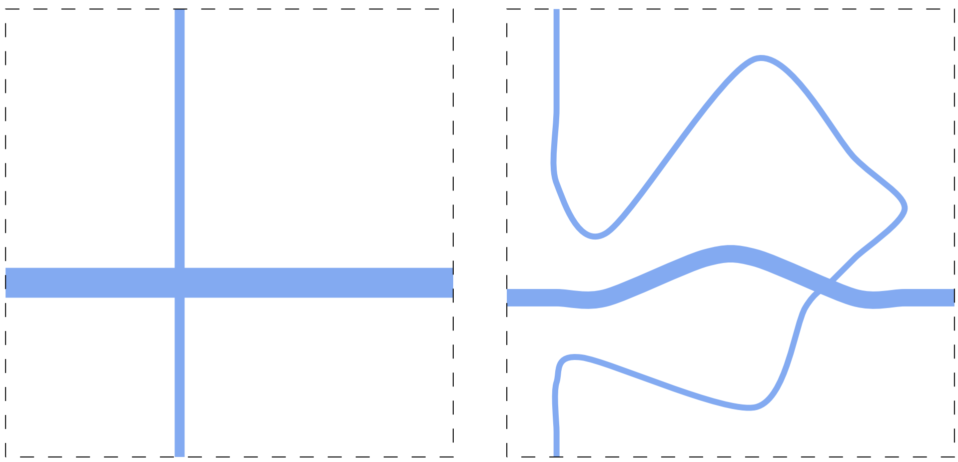


Figure 1: Two of the various possibilities for localizing the controls. In each picture, the (blue) filled part illustrates a valid control region $\Omega \subset \mathbb{T}^2$, which can be taken of arbitrary nonzero area.

1.1 The controllability problem

Let a viscous Newtonian fluid of velocity $\boldsymbol{u}\colon \mathbb{T}^2 \times (0, T_{\mathrm{ctrl}}) \longrightarrow \mathbb{R}^2$ and exerted pressure $p\colon \mathbb{T}^2 \times (0, T_{\mathrm{ctrl}}) \longrightarrow \mathbb{R}$ be governed by the two-dimensional incompressible Navier–Stokes system

$$\partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{f} + \mathbb{I}_{\Omega}\boldsymbol{\xi}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \quad \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0, \tag{1.1}$$

where $\nu > 0$ quantifies the viscosity of the fluid at hand, $\boldsymbol{u}_0\colon \mathbb{T}^2 \longrightarrow \mathbb{R}^2$ stands for the initial velocity, $\boldsymbol{f}\colon \mathbb{T}^2 \times (0, T_{\mathrm{ctrl}}) \longrightarrow \mathbb{R}^2$ is a known external force, $\mathbb{I}_{\Omega}$ denotes the indicator function of the subset Ω, and $\boldsymbol{\xi}\colon \mathbb{T}^2 \times (0, T_{\mathrm{ctrl}}) \longrightarrow \mathbb{R}^2$ represents a control sought to approximately steer the fluid in any fixed time $T_{\mathrm{ctrl}} > 0$ towards any prescribed admissible target state $\boldsymbol{u}_1\colon \mathbb{T}^2 \longrightarrow \mathbb{R}^2$. The system (1.1) is said to be globally approximately controllable if for any given states $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ belonging to a certain space with norm $\|\cdot\|$ and arbitrarily chosen accuracy parameter $\varepsilon > 0$, there exists a control $\boldsymbol{\xi}$ such that the corresponding solution to (1.1) satisfies

$$\|\boldsymbol{u}(\cdot, T_{\mathrm{ctrl}}) - \boldsymbol{u}_1\| < \varepsilon.$$

In view of applications where only a limited number of control actions can be realized, and where explicit representations of the controls are required, it would be desirable to achieve the global approximate controllability of (1.1) by means of finite-dimensional controls

$$\boldsymbol{\xi}(\boldsymbol{x},t) = \alpha_1(t)\boldsymbol{\psi}_1(\boldsymbol{x}) + \cdots + \alpha_N(t)\boldsymbol{\psi}_N(\boldsymbol{x}), \tag{1.2}$$

where $\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_N\colon \mathbb{T}^2 \longrightarrow \mathbb{R}^2$ are linearly independent vector fields unrelated to the prescribed data and the viscosity. This task amounts to identifying a fixed number of parameters $\alpha_1(t), \ldots, \alpha_N(t)$. When $\Omega = \mathbb{T}^2$, the celebrated Agrachev–Sarychev approach [2] and its various advancements allow solving problems of this type. In reality, however, designing controls to act in the whole domain is difficult to justify. Therefore, the spotlight is put on the challenging case $\Omega \neq \mathbb{T}^2$. In fact, the construction of physically localized and finite-dimensional controls constitutes a widely open problem described by Agrachev in [1, Section 7].

The approach proposed in the present article can be viewed as a step towards this open problem in the following sense. We explicitly fix nine effectively constructed functions

$$\boldsymbol{\vartheta}_1, \ldots, \boldsymbol{\vartheta}_4\colon \mathbb{T}^2 \times (0,1) \longrightarrow \mathbb{R}^2, \quad \boldsymbol{\vartheta}_5, \ldots, \boldsymbol{\vartheta}_9\colon \mathbb{T}^2 \longrightarrow \mathbb{R}^2, \quad \bigcup_{l=1}^{9} \operatorname{supp}(\boldsymbol{\vartheta}_l) \subset \Omega$$

and subsequently seek the control $\boldsymbol{\xi}\colon \mathbb{T}^2 \times (0, T_{\mathrm{ctrl}}) \longrightarrow \mathbb{R}^2$ of the specific form

$$\boldsymbol{\xi}(\boldsymbol{x},t) = \sum_{l=1}^{4} \gamma_l(t)\boldsymbol{\vartheta}_l(\boldsymbol{x}, \sigma(T_{\mathrm{ctrl}} - t)) + \sum_{l=5}^{9} \gamma_l(t)\boldsymbol{\vartheta}_l(\boldsymbol{x}). \tag{1.3}$$

In order to provide a more precise breakdown how the driving force $\boldsymbol{\xi}$ will be localized, we introduce any fixed nonempty subdomain $\omega \subset \Omega$ and choose $\boldsymbol{\vartheta}_1, \ldots, \boldsymbol{\vartheta}_9$ with

$$\bigcup_{l=1}^{7} \operatorname{supp}(\boldsymbol{\vartheta}_l) \subset \omega, \quad \operatorname{supp}(\boldsymbol{\vartheta}_8) \cup \operatorname{supp}(\boldsymbol{\vartheta}_9) \subset \Omega.$$

The known profiles $\boldsymbol{\vartheta}_1, \ldots, \boldsymbol{\vartheta}_9$ only depend on the fixed control region Ω and the chosen subdomain ω. In particular, they are independent of the prescribed initial state, the target state, the viscosity, the force $\boldsymbol{f}$, and the control time T_{ctrl}. In this way, the global approximate controllability for the system (1.1) shall be established by identifying the parameters $\gamma_1(t), \ldots, \gamma_9(t)$ and the scaling constant $\sigma \geq T_{\mathrm{ctrl}}^{-1}$, depending on the given data (*cf.* Figure 2). In fact, σ can be any sufficiently large number and we are able to express $\gamma_5, \ldots, \gamma_9$ by means of $\gamma_1, \ldots, \gamma_4$ through universal formulas (*cf.* (2.12)), which justifies the point of view that one merely has to act on the system (1.1) with four controls:

$$\gamma_1, \ldots, \gamma_4 \in \mathrm{L}^2((0, T_{\mathrm{ctrl}}); \mathbb{R}).$$

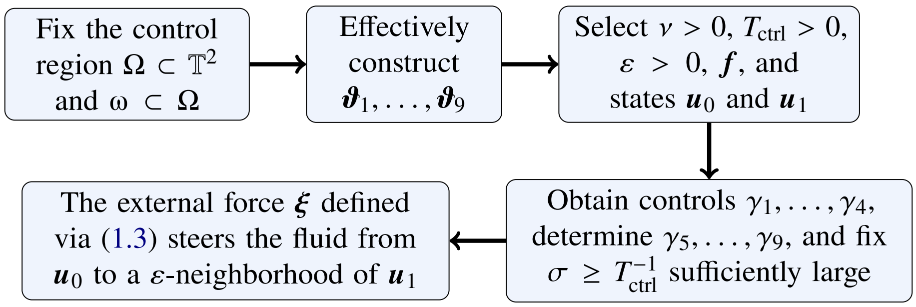


Figure 2: An illustration of the dependencies in the proposed framework.

1.2 Notations

The basic L^2-based spaces of average-free scalar fields and divergence-free vector fields are specified by

$$\mathrm{H}_{avg} := \left\{ f \in \mathrm{L}^2(\mathbb{T}^2; \mathbb{R}) \mid \int_{\mathbb{T}^2} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = 0 \right\}, \quad \mathbf{H}_{div} := \left\{ \boldsymbol{f} \in \mathrm{L}^2(\mathbb{T}^2; \mathbb{R}^2) \mid \boldsymbol{\nabla} \cdot \boldsymbol{f} = 0 \right\}.$$

Moreover, for any $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we employ the function spaces

$$\mathrm{H}^m := \mathrm{H}^m(\mathbb{T}^2; \mathbb{R}) \cap \mathrm{H}_{avg}, \quad \mathbf{V}^m := \mathrm{H}^m(\mathbb{T}^2; \mathbb{R}^2) \cap \mathbf{H}_{div},$$

where the L^2-based Sobolev spaces $\mathrm{H}^m(\mathbb{T}^2; \mathbb{R})$ and $\mathrm{H}^m(\mathbb{T}^2; \mathbb{R}^2)$ are equipped with the canonical inner product $\langle \cdot, \cdot \rangle_m$ and the induced norm $\| \cdot \|_m$. Furthermore, the standard basis vectors of $\mathbb{R}^2$ are denoted by $\boldsymbol{e}_1 := [1, 0]^\top$ and $\boldsymbol{e}_2 := [0, 1]^\top$.

1.3 Main result

Let an arbitrary integer $N \geq 4$ be fixed throughout. As a generalization of (1.3), we consider a localized force $\boldsymbol{\xi} = \boldsymbol{\xi}_{\gamma_1,\ldots,\gamma_N,\sigma}$ of the form

$$\boldsymbol{\xi}(\boldsymbol{x},t) = \sum_{l=1}^{N} \gamma_l(t)\boldsymbol{\vartheta}_l(\boldsymbol{x},\sigma(T_{\mathrm{ctrl}}-t)) + \sum_{l=N+1}^{N+5} \gamma_l(t)\boldsymbol{\vartheta}_l(\boldsymbol{x}) \tag{1.4}$$

consisting of

- known effectively constructed functions (*cf.* Sections 2 and 3)

$$\boldsymbol{\vartheta}_1,\ldots,\boldsymbol{\vartheta}_N \colon \mathbb{T}^2 \times (0,1) \longrightarrow \mathbb{R}^2, \quad \boldsymbol{\vartheta}_{N+1},\ldots,\boldsymbol{\vartheta}_{N+5} \colon \mathbb{T}^2 \longrightarrow \mathbb{R}^2,$$
$$\bigcup_{l=1}^{N+3} \mathrm{supp}(\boldsymbol{\vartheta}_l) \subset \omega, \quad \mathrm{supp}(\boldsymbol{\vartheta}_{N+4}) \cup \mathrm{supp}(\boldsymbol{\vartheta}_{N+5}) \subset \Omega,$$

- unknown controls $\gamma_1,\ldots,\gamma_{N+5} \in \mathrm{L}^2((0,T_{\mathrm{ctrl}});\mathbb{R})$ and $\sigma \geq T_{\mathrm{ctrl}}^{-1}$, where $\gamma_{N+1},\ldots,\gamma_{N+5}$ can be expressed by means of $\gamma_1,\ldots,\gamma_N$.

We will provide a family of possibilities for selecting $\boldsymbol{\vartheta}_1,\ldots,\boldsymbol{\vartheta}_{N+5}$. Intuitively, the functions $\boldsymbol{\vartheta}_1,\ldots,\boldsymbol{\vartheta}_N$ correspond to the Fourier modes that are allowed to be triggered by the controls. The additional profiles $\boldsymbol{\vartheta}_{N+1},\boldsymbol{\vartheta}_{N+2},\boldsymbol{\vartheta}_{N+3}$ arise from average corrections performed at the vorticity level, while the vector fields $\boldsymbol{\vartheta}_{N+4},\boldsymbol{\vartheta}_{N+5}$ are involved in regulating the velocity average. Notably, resting on the assumption that Ω contains two suitable cuts, $\boldsymbol{\vartheta}_{N+4}$ and $\boldsymbol{\vartheta}_{N+5}$ are chosen curl-free.

Theorem 1.1 (Main result). *Let $r \in \mathbb{N}$ with $r \geq 2$ and fix $T_{\mathrm{ctrl}} > 0$, $\boldsymbol{u}_0, \boldsymbol{u}_1 \in \mathbf{V}^r$, $\boldsymbol{f} \in \mathrm{L}^2((0,T_{\mathrm{ctrl}});\mathrm{H}^r(\mathbb{T}^2;\mathbb{R}^2))$, and $\varepsilon > 0$. There exist a scaling constant $\sigma \geq T_{\mathrm{ctrl}}^{-1}$ and $N+5$ control parameters*

$$\gamma_1,\ldots,\gamma_{N+5} \in \mathrm{L}^2((0,T_{\mathrm{ctrl}});\mathbb{R})$$

such that the unique solution $\boldsymbol{u} \in \mathrm{C}^0([0,T_{\mathrm{ctrl}}];\mathbf{V}^r) \cap \mathrm{L}^2((0,T_{\mathrm{ctrl}});\mathbf{V}^{r+1})$ to the Navier–Stokes problem (1.1) with the external force $\boldsymbol{\xi}$ from (1.4) satisfies the terminal condition

$$\|\boldsymbol{u}(\cdot,T_{\mathrm{ctrl}}) - \boldsymbol{u}_1\|_r < \varepsilon.$$

Moreover, the coefficients $\gamma_{N+1},\ldots,\gamma_{N+5}$ are determined from $\gamma_1,\ldots,\gamma_N$ through explicit formulas.

To our knowledge, Theorem 1.1 constitutes the first result in the direction of Agrachev's open problem posed in [1, Section 7]. However, strictly speaking, the original version of the latter problem is not fully resolved in this article. Indeed, our fixed functions $\boldsymbol{\vartheta}_1,\ldots,\boldsymbol{\vartheta}_N$ naturally depend on the time variable, while the formulation of the problem raised in [1] asks for a control of the type (1.2) with time-independent modes $\boldsymbol{\psi}_1,\ldots,\boldsymbol{\psi}_N$.

1.4 Methodology

Our strategy for showing Theorem 1.1 is newly developed and promotes viewing the profiles $\boldsymbol{\vartheta}_1, \ldots, \boldsymbol{\vartheta}_N$ from (1.4) as "transported Fourier modes". This approach comprises explicit ad-hoc constructions and the following main ingredients:

- the return method introduced by Coron in [6] for the stabilization of a mechanical system, and which has thereafter been applied to numerous nonlinear partial differential equations (*cf.* [9, Part 2, Chapter 6]);
- a linear test for approximate controllability developed in [17], and which involves the notion of observable vector fields introduced in [13];
- a convection strategy on the torus by virtue of rigid translations, based on a special covering of $\mathbb{T}^2$ using overlapping squares (*cf.* Theorem 3.3);
- a simplified saturation property without length condition (*cf.* Section 4);
- average corrections with memory (*cf.* Remark 5.4).

The controllability problem for the velocity is hereby reduced to one for the vorticity, which then allows to utilize the underlying transport mechanisms of the considered fluid model. We linearize the vorticity equation around a special return method profile and derive a related controllability problem for a scalar transport equation. However, in order to determine the parameters $\gamma_1, \ldots, \gamma_{N+5}$ in (1.4), we have to consider a modified transport problem where convection takes place along a vector field encoding a certain observability property. Concerning the aforementioned setup, we obtain finite-dimensional controls of the type

$$g = \sum_{\boldsymbol{\ell} \in \mathcal{K}} (\zeta^s_{\boldsymbol{\ell}}(t) \sin(\boldsymbol{\ell} \cdot \boldsymbol{x}) + \zeta^c_{\boldsymbol{\ell}}(t) \cos(\boldsymbol{\ell} \cdot \boldsymbol{x})), \tag{1.5}$$

where the pairs of integers $\boldsymbol{\ell} = (\ell_1, \ell_2)$ are taken from a finite family $\mathcal{K} \subset \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ and the set $\mathcal{K}$ provides the Fourier modes which are allowed to be triggered by the controls. The coefficients $(\zeta^s_{\boldsymbol{\ell}}, \zeta^c_{\boldsymbol{\ell}})$ in (1.5), which will be used to calculate $\gamma_1, \ldots, \gamma_{N+5}$, depend only on time and can be viewed as the true controls. Subsequently, the action of the force g is squeezed into the control region ω. To this end, the background flow encoding the observability property is interchanged with a return method trajectory shifting the whole torus $\mathbb{T}^2$ until each particle has passed through ω. The latter step will hereby motivate the particular constructions of the building blocks $\boldsymbol{\vartheta}_1, \ldots, \boldsymbol{\vartheta}_N$, which are essentially the Fourier modes from (1.5) composed with certain flow maps. The profiles $\boldsymbol{\vartheta}_{N+1}, \boldsymbol{\vartheta}_{N+2}, \boldsymbol{\vartheta}_{N+3}$ ensure that the force acting on the vorticity is average-free. Two more functions $\boldsymbol{\vartheta}_{N+4}, \boldsymbol{\vartheta}_{N+5}$, which shall be curl-free, are then required to also act on the velocity average. Next, scaling arguments related to the return method yield the approximate controllability in a short time by means of large controls for the original nonlinear vorticity problem. Eventually, the proof of Theorem 1.1 is concluded by deriving formulas for the velocity controls based on the previously found vorticity controls.

1.5 Related literature

There exists a rich literature on controllability problems for incompressible Navier–Stokes and Euler systems, or other related models. A traditional point of view has been to seek the controls as abstract elements of infinite-dimensional function spaces, with significant attention being devoted to the case of bounded domains. Typically, for such a setup, one aims to obtain interior controls localized within a small subdomain or controls acting on a small part of the domain's boundary. In this context, J.-L. Lions [15] raised several open problems which have so far inspired more than three decades of fruitful research, while key questions such as the global approximate controllability for the Navier–Stokes system with the no-slip boundary condition remain open until the present day. Substantial milestones have been accomplished by Coron in [7, 8], where he obtains by way of his return method the global exact and approximate controllability for planar Euler and Navier–Stokes problems respectively. Glass further developed the return method in [12] to address three-dimensional configurations. A different point of view has been pursued by Lions and Zuazua in [16], where they achieved exact controllability results for Galerkin's approximations of incompressible Navier–Stokes problems by combining duality arguments with a contraction mapping principle. Several recent advances regarding the incompressible Navier–Stokes system are due to Coron *et al.* in [10] and Liao *et al.* in [14], noting that this list is far away from being comprehensive and that many past, as well as contemporary, developments are well captured by the references therein.

The question of controllability by finite-dimensional controls supported in the entire domain constitutes a subject of active research, as well. In this case, the controls are sought to be of a very specific form (*cf.* (1.2)), but are, so far, not physically localized in a given subdomain. A major breakthrough in this direction has been achieved for two-dimensional periodic domains by Agrachev and Sarychev in [2–4], who developed a geometric control approach that has subsequently been extended and improved in various ways. For instance, drawing upon the geometric arguments due to Agrachev and Sarychev, three-dimensional Navier–Stokes problems have been treated by Shirikyan in [20, 21] and perfect fluids by Nersisyan in [18]. A new proof for results of the same type has been presented recently by Nersesyan in [17], where Coron's return method is employed in combination with a special linear test. Moreover, Phan and Rodrigues consider in [19] a Navier–Stokes problem which is posed in a cubical domain with the Lions boundary conditions. A concise and self-contained account of the Agrachev–Sarychev method, elaborating the example of the one-dimensional Burgers equation, is provided in [22].

So far, the previous two paragraphs resemble rather disjoint lines of research. The notion of transported Fourier modes, which are the building blocks of our control force, is in this regard intended to connect concepts from both worlds. At least three immediate questions remain. Is it possible to build $\boldsymbol{\vartheta}_1, \ldots, \boldsymbol{\vartheta}_N$ independent of time? Can one dispense with the profiles $\boldsymbol{\vartheta}_{N+4}$ and $\boldsymbol{\vartheta}_{N+5}$ that are here necessarily supported along smooth cuts? How to extend the presented method in order to accommodate flows past physical boundaries?

1.6 Organization of this article

In Section 2, the proof of Theorem 1.1 is reduced to the resolution of a controllability problem for the vorticity. In Section 3, the constructions of $\boldsymbol{\vartheta}_1, \ldots, \boldsymbol{\vartheta}_{N+3}$ are completed. Then, a saturation property is characterized in Section 4. Finally, the key steps from the proof of Theorem 1.1 are carried out in Section 5. Namely, the control parameters are obtained in Section 5.1, the localization arguments are developed in Section 5.2, while Section 5.3 explains the passage to the nonlinear problem. An appendix is concerned with the description of certain cutoff vector fields used for defining the velocity average controls $\boldsymbol{\vartheta}_{N+4}, \boldsymbol{\vartheta}_{N+5}$.

2 Proof of the main result unfolded

This section provides the detailed road map for this article. First, a vorticity formulation of (1.1) is presented, accompanied by a well-posedness result in Section 2.1. Next, the curled versions of the profiles $\boldsymbol{\vartheta}_1, \ldots, \boldsymbol{\vartheta}_{N+3}$ from (1.4) are described in Section 2.2, noting that $\boldsymbol{\vartheta}_{N+4}$ and $\boldsymbol{\vartheta}_{N+5}$ will be chosen curl-free. Afterwards, in the course of Section 2.3, a version of Theorem 1.1 for the Navier–Stokes system in vorticity form is established. This constitutes the main part of this work and several steps are outsourced to subsequent sections. Ultimately, the proof of Theorem 1.1 is completed in Section 2.4.

We write $\boldsymbol{\nabla} \wedge \boldsymbol{g} \coloneqq \partial_1 g_2 - \partial_2 g_1$ for the curl of a vector field $\boldsymbol{g} = [g_1, g_2]^\top$ and denote by $r \in \mathbb{N}$ the regularity parameter from Theorem 1.1. Moreover, we remind that $\omega \subset \Omega$ represents the small subregion fixed in the beginning. As previously mentioned, the goal is now to control the time evolution of the vortex

$$w = \boldsymbol{\nabla} \wedge \boldsymbol{u} \colon \mathbb{T}^2 \times (0, T_{\mathrm{ctrl}}) \longrightarrow \mathbb{R}$$

which solves in $\mathbb{T}^2 \times (0, T_{\mathrm{ctrl}})$ the incompressible vorticity problem

$$\begin{cases} \partial_t w - \nu \Delta w + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\, w = h + \mathbb{I}_{\omega} \eta, \\ \boldsymbol{\nabla} \wedge \boldsymbol{u} = w, \quad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \quad \fint_{\mathbb{T}^2} \boldsymbol{u}(\boldsymbol{x}, \cdot)\, \mathrm{d}\boldsymbol{x} = \boldsymbol{\aleph}, \\ w(\cdot, 0) = w_0, \end{cases} \tag{2.1}$$

where the control η satisfies $\operatorname{supp}(\eta) \subset \omega$ and is sought to ensure the terminal condition

$$\|w(\cdot, T_{\mathrm{ctrl}}) - w_1\|_{r-1} < \varepsilon. \tag{2.2}$$

Furthermore, the controllability problem specified by (2.1) and (2.2) involves

- the prescribed states $w_0 \coloneqq \boldsymbol{\nabla} \wedge \boldsymbol{u}_0$ and $w_1 \coloneqq \boldsymbol{\nabla} \wedge \boldsymbol{u}_1$,
- the known external force $h \coloneqq \boldsymbol{\nabla} \wedge \boldsymbol{f} \colon \mathbb{T}^2 \times [0, T_{\mathrm{ctrl}}] \to \mathbb{R}$,
- the fixed approximation accuracy $\varepsilon > 0$ and chosen control time $T_{\mathrm{ctrl}} > 0$,

along with

- the velocity average contribution $\aleph = \aleph_\sigma$, which belongs to $\mathrm{W}^{1,2}((0,1);\mathbb{R}^2)$ and is given in terms of the yet unknown scaling parameter $\sigma > 0$ by means of the formula

$$\begin{aligned}\aleph_\sigma(t) &\coloneqq \mathbb{I}_{[0,T_\sigma]}(t)\int_{\mathbb{T}^2} \boldsymbol{u}_0(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \mathbb{I}_{[T_\sigma,T_{\mathrm{ctrl}}]}(t)(\sigma\overline{\boldsymbol{y}}+\boldsymbol{U})(\sigma(t-T_\sigma)),\\ T_\sigma &\coloneqq T_{\mathrm{ctrl}} - \sigma^{-1},\end{aligned} \tag{2.3}$$

  where the profile $\overline{\boldsymbol{y}} \in \mathrm{C}_0^\infty((0,1);\mathbb{R}^2)$ in (2.3) only depends on ω and is explicitly constructed in Section 3.3, while $\boldsymbol{U} \in \mathrm{W}^{1,2}((0,1);\mathbb{R}^2)$ is selected such that

$$\forall t \in [0,1/4]\colon\ \boldsymbol{U}(t) = \int_{\mathbb{T}^2} \boldsymbol{u}_0(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \quad \forall t \in [3/4,1]\colon\ \boldsymbol{U}(1) = \int_{\mathbb{T}^2} \boldsymbol{u}_1(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

- the control force $\eta = \eta_{\gamma_1,\dots,\gamma_N,\sigma} \coloneqq \boldsymbol{\nabla}\wedge\boldsymbol{\xi}$, which shall be explicitly described up to the yet unknown coefficients $\gamma_1, \dots, \gamma_N$ and $\sigma \geq T_{\mathrm{ctrl}}^{-1}$ (*cf.* Section 2.2).

2.1 Well-posedness

For any $m \in \mathbb{N}$, let $\boldsymbol{\Upsilon}\colon \mathrm{H}^m \times \mathbb{R}^2 \longrightarrow \mathbf{V}^{m+1}$ be the operator that assigns to $z \in \mathrm{H}^m$ and $\boldsymbol{A} \in \mathbb{R}^2$ the unique vector field $\boldsymbol{g} = \boldsymbol{\Upsilon}(z,\boldsymbol{A}) \in \mathbf{V}^{m+1}$ satisfying

$$\boldsymbol{\nabla}\wedge\boldsymbol{g} = z, \quad \boldsymbol{\nabla}\cdot\boldsymbol{g} = 0, \quad \int_{\mathbb{T}^2} \boldsymbol{g}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \boldsymbol{A}. \tag{2.4}$$

The operator $\boldsymbol{\Upsilon}$ can be expressed by means of the formula

$$\boldsymbol{\Upsilon}(z,\boldsymbol{A}) \coloneqq \begin{bmatrix} \partial_2\varphi \\ -\partial_1\varphi \end{bmatrix} + \frac{1}{\int_{\mathbb{T}^2}\mathrm{d}\boldsymbol{x}}\boldsymbol{A},$$

where the stream function $\varphi \in \mathrm{H}^{m+2}$ is the unique solution to the Poisson problem

$$-\Delta\varphi = z, \quad \int_{\mathbb{T}^2} \varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0.$$

We proceed with a few remarks concerning the global well-posedness of the two-dimensional Navier–Stokes system. Here, for any given time $T > 0$ and a regularity parameter $m \in \mathbb{N}$, the solution space for the vorticity is chosen as

$$\mathcal{X}_T^m \coloneqq \mathrm{C}^0([0,T];\mathrm{H}^m) \cap \mathrm{L}^2((0,T);\mathrm{H}^{m+1}),$$

endowed with the canonical norm

$$\|\cdot\|_{\mathcal{X}_T^m} \coloneqq \|\cdot\|_{\mathrm{C}^0([0,T];\mathrm{H}^m)} + \|\cdot\|_{\mathrm{L}^2((0,T);\mathrm{H}^{m+1})}.$$

Lemma 2.1. *For any $w_0 \in \mathrm{H}^m$, $\boldsymbol{A} \in \mathrm{W}^{1,2}((0,T);\mathbb{R}^2)$, and $h \in \mathrm{L}^2((0,T);\mathrm{H}^{m-1})$, there exists a unique solution $w = S_T(w_0, h, \boldsymbol{A}) \in \mathcal{X}_T^m$ to the vorticity equation*

$$\partial_t w - \nu\Delta w + (\boldsymbol{\Upsilon}(w, \boldsymbol{A}) \cdot \boldsymbol{\nabla})\, w = h, \quad w(\cdot, 0) = w_0, \tag{2.5}$$

where the resolving operator for (2.5) is denoted by

$$S_T(\cdot,\cdot,\cdot)\colon \mathrm{H}^m \times \mathrm{L}^2((0,T);\mathrm{H}^{m-1}) \times \mathrm{W}^{1,2}((0,T);\mathbb{R}^2) \longrightarrow \mathcal{X}_T^m, \quad (w_0, h, \boldsymbol{A}) \longmapsto w.$$

Proof. Given any force $\boldsymbol{f} \in \mathrm{L}^2((0,T);\mathbf{V}^m)$ with $\boldsymbol{\nabla} \wedge \boldsymbol{f} = h$ and initial data $\boldsymbol{u}_0 := \boldsymbol{\Upsilon}(w_0, \boldsymbol{A}(0)) \in \mathbf{V}^{m+1}$, one may reduce the existence and regularity statements of Lemma 2.1 to a corresponding one for the two-dimensional Navier–Stokes problem in velocity form

$$\partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\, \boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{f} + \partial_t \boldsymbol{a}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \quad \boldsymbol{u}(\cdot, 0) = \boldsymbol{u}_0, \tag{2.6}$$

where the function

$$\boldsymbol{a}(t) := \boldsymbol{A}(t) - \int_0^t \int_{\mathbb{T}^2} \boldsymbol{f}(\boldsymbol{x}, s)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}s$$

is chosen such that the average of the solution $\boldsymbol{u}(t)$ to (2.6) is equal to $\boldsymbol{A}(t)$. Regarding the well-posedness of (2.6), we refer, for instance, to [5, 23]. The uniqueness of solutions to (2.5) in the space $\mathcal{X}_T^m$ is a consequence of the usual energy estimates. □

2.2 A brief description of the controls

We further specify our ansatz for the control force $\eta = \eta_{\gamma_1,\dots,\gamma_N,\sigma}$ in (2.1). Hereto, it can be assumed, without loss of generality, that the integer N is even, as otherwise the case $N-1$ may be considered instead. Moreover, a family of trigonometric functions is denoted by

$$s_{\boldsymbol{\ell}}(\boldsymbol{x}) := \sin(\boldsymbol{\ell} \cdot \boldsymbol{x}), \quad c_{\boldsymbol{\ell}}(\boldsymbol{x}) := \cos(\boldsymbol{\ell} \cdot \boldsymbol{x}), \quad \boldsymbol{\ell} \in \mathbb{Z}_*^2, \tag{2.7}$$

where $\mathbb{Z}_*^2 := \mathbb{Z}^2 \setminus \{\mathbf{0}\}$. We continue by selecting a subset $\mathcal{K} \subset \mathbb{Z}_*^2$ of cardinality $N/2$ such that $\mathbb{Z}^2 = \mathrm{span}_{\mathbb{Z}}(\mathcal{K})$. Subsequently, we perform a renaming of the type $(\gamma_{\boldsymbol{\ell}}^s)_{\boldsymbol{\ell}\in\mathcal{K}} = (\gamma_j)_{j\in\{1,\dots,N/2\}}$ and $(\gamma_{\boldsymbol{\ell}}^c)_{\boldsymbol{\ell}\in\mathcal{K}} = (\gamma_j)_{j\in\{N/2+1,\dots,N\}}$, followed by proposing for the force η an ansatz of the form

$$\eta(\boldsymbol{x}, t) := \sigma \mathbb{I}_{[T_\sigma, T_{\mathrm{ctrl}}]}(t) \widetilde{\eta}(\boldsymbol{x}, \sigma(t - T_\sigma)), \tag{2.8}$$

where the auxiliary profile $\widetilde{\eta}$ will be introduced below, $\sigma \geq T_{\mathrm{ctrl}}^{-1}$ depends on the prescribed data, $T_\sigma = T_{\mathrm{ctrl}} - \sigma^{-1}$ is as in (2.3), and the presence of the indicator function $\mathbb{I}_{[T_\sigma, T_{\mathrm{ctrl}}]}$ in (2.8) signals that the controls are only active during a short time interval of length σ^{-1}.

Before presenting the detailed ansatz for the force $\widetilde{\eta}$ appearing in (2.8), let us outline the underlying motivations. At this point, we foreshadow the analysis of Sections 3-5 and assume that certain control parameters

$$(\zeta^s_{\boldsymbol{\ell}})_{\boldsymbol{\ell}\in\mathcal{K}} = (\zeta_j)_{j\in\{1,\dots,N/2\}}, \quad (\zeta^c_{\boldsymbol{\ell}})_{\boldsymbol{\ell}\in\mathcal{K}} = (\zeta_j)_{j\in\{N/2+1,\dots,N\}}$$

have already been retrieved by solving a controllability problem for a linear transport equation driven by a force of the type (1.5). We then define a physically localized control force $\widehat{\eta}\colon \mathbb{T}^2 \times [0,1] \longrightarrow \mathbb{R}$ of the form

$$\widehat{\eta}(\boldsymbol{x},t) \coloneqq \chi(\boldsymbol{x}) \sum_{j=1}^{K} \mathbb{I}_{[t^j_a,t^j_b]}(t) \sum_{\boldsymbol{\ell}\in\mathcal{K}} \left[\zeta^s_{\boldsymbol{\ell}}(\tau_j(t)) s_{\boldsymbol{\ell}}\left(\Xi(\boldsymbol{x},t)\right) + \zeta^c_{\boldsymbol{\ell}}(\tau_j(t)) c_{\boldsymbol{\ell}}\left(\Xi(\boldsymbol{x},t)\right)\right], \tag{2.9}$$

which shall become meaningful in the course of Section 5.2. In (2.9), anticipating the constructions of Section 3, the cutoff χ is supported in ω, the square number $K \in \mathbb{N}$ determines a covering of the torus by squares smaller than ω, the flow map Ξ is essentially responsible for squeezing the actions of global controls into the region ω, $(\tau_l\colon [0,1] \longrightarrow [0,1])_{l\in\{1,\dots,K\}}$ is a family of parameter changes, and the numbers

$$0 < t^1_a < t^1_b < t^2_a < \cdots < t^K_b < 1, \quad T^\star = t^l_a - t^l_b, \quad 3T^\star = t^{l+1}_a - t^l_a$$

give rise to a partition of the reference time interval $[0,1]$. A key problem, however, is that we are unable to ensure the expression in (2.9) to have zero average, which renders the hypothetical choice $\widetilde{\eta} = \widehat{\eta}$ unsuitable. In order to transship this issue, we are going to put in place special smooth cutoff functions $\widetilde{\chi}$ and $(\widetilde{\chi}_l)_{l\in\{1,\dots,K\}}$ supported in ω, noting that the family $(\widetilde{\chi}_l)_{l\in\{1,\dots,K\}}$ shall only consist of the two different elements $\widetilde{\chi}_{\nearrow}$ and $\widetilde{\chi}_{\rightarrow}$, in the sense that (*cf.* (3.5))

$$\widetilde{\chi}_l = \begin{cases} \widetilde{\chi}_{\nearrow} = \text{top-right shift of } \widetilde{\chi}, & \text{if } l \text{ is a multiple of } \sqrt{K},\\ \widetilde{\chi}_{\rightarrow} = \text{right shift of } \widetilde{\chi}, & \text{otherwise.} \end{cases}$$

We then define $\widetilde{\eta}$ in (2.8) as the zero-average version of $\widehat{\eta}$ given by

$$\begin{aligned} \widetilde{\eta}(\boldsymbol{x},t) &= \widetilde{\eta}_{\zeta_1,\dots,\zeta_N}(\boldsymbol{x},t)\\ &\coloneqq \widehat{\eta}(\boldsymbol{x},t) - \sum_{i=1}^{K}\sum_{j=1}^{i} \mathbb{I}_{[t^i_a,t^i_b]}(t)\widetilde{\chi}_i(\boldsymbol{x}) \int_{\mathbb{T}^2} \widehat{\eta}(z, t-3(j-1)T^\star)\,\mathrm{d}z\\ &\quad + \sum_{i=2}^{K}\sum_{k=1}^{i-1} \mathbb{I}_{[t^i_a,t^i_b]}(t)\widetilde{\chi}(\boldsymbol{x}) \int_{\mathbb{T}^2} \widehat{\eta}(z, t-3kT^\star)\,\mathrm{d}z, \end{aligned} \tag{2.10}$$

which might appear artificially sophisticated at first, yet ensures that $\widetilde{\eta}$ constitutes a suitable control. In view of the arguments carried out in Section 5.2, where the characteristic curves of certain controlled transport problems are studied, the definition in (2.10) can be interpreted as the realization of an average correction strategy with memory (*cf.* Remark 5.4). The parcels passing through the control region, when being transported along an appropriate vector field, are divided into several groups. Some are controlled as desired, while others are asked to further carry around the average corrections until visiting the control region again. In order to see that $\widetilde{\eta}$ defined via (2.10) is average-free, we anticipate that the cutoff functions $\widetilde{\chi}$ and $\widetilde{\chi}_i$ shall be of average 1 (*cf.* Section 3.2). Accordingly, when integrating in (2.10) over $\mathbb{T}^2$ and accounting for various cancellations, one observes

$$\begin{aligned}\int_{\mathbb{T}^2} \widetilde{\eta}(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{T}^2} \widehat{\eta}(z,t)\,\mathrm{d}z - \sum_{i=1}^{K}\sum_{j=1}^{i} \mathbb{I}_{[t_a^i,t_b^i]}(t)\int_{\mathbb{T}^2} \widehat{\eta}(z,t-3(j-1)T^{\star})\,\mathrm{d}z \\ &\quad + \sum_{i=2}^{K}\sum_{k=1}^{i-1} \mathbb{I}_{[t_a^i,t_b^i]}(t)\int_{\mathbb{T}^2} \widehat{\eta}(z,t-3kT^{\star})\,\mathrm{d}z \\ &= 0.\end{aligned}$$

In summary, the control force η assumes the form

$$\eta(\boldsymbol{x},t) = \sum_{l=1}^{N} \gamma_l(t)\eta_l(\boldsymbol{x},\sigma(t-T_\sigma)) + \sum_{l=N+1}^{N+3} \gamma_l(t)\eta_l(\boldsymbol{x}), \tag{2.11}$$

where

$$\gamma_l(t) := \sigma\mathbb{I}_{[T_\sigma,T_{\mathrm{ctrl}}]}(t)\widetilde{\gamma}_l(\sigma(t-T_\sigma)), \tag{2.12}$$

with

$$\widetilde{\gamma}_l(t) := \begin{cases} \sum_{j=1}^{K} \mathbb{I}_{[t_a^j,t_b^j]}(t)\zeta_l(\tau_j(t)), & l \leq N, \\ \sum_{i=2}^{K}\sum_{k=1}^{i-1} \mathbb{I}_{[t_a^i,t_b^i]}(t)\int_{\mathbb{T}^2} \widehat{\eta}(z,t-3kT^{\star})\,\mathrm{d}z, & l = N+1, \\ -\sum_{i=1}^{\sqrt{K}}\sum_{j=1}^{i\sqrt{K}} \mathbb{I}_{[t_a^{i\sqrt{K}},t_b^{i\sqrt{K}}]}(t)\int_{\mathbb{T}^2} \widehat{\eta}(z,t-3(j-1)T^{\star})\,\mathrm{d}z, & l = N+2, \\ -\gamma_{N+2}(t) - \sum_{i=1}^{K}\sum_{j=1}^{i} \mathbb{I}_{[t_a^i,t_b^i]}(t)\int_{\mathbb{T}^2} \widehat{\eta}(z,t-3(j-1)T^{\star})\,\mathrm{d}z, & l = N+3. \end{cases} \tag{2.13}$$

Inserting (2.9) through (2.13) into (2.12) provides hereby formulas for γ_{N+1}, γ_{N+2}, and γ_{N+3} in terms of $\gamma_1,\ldots,\gamma_N$. Furthermore, owing to the previously fixed notations $(s_\ell)_{\ell\in\mathcal{K}} = (s_j)_{j\in\{1,\ldots,N/2\}}$ and $(c_\ell)_{\ell\in\mathcal{K}} = (c_j)_{j\in\{N/2+1,\ldots,N\}}$, the modes η_l in (2.12) are explicitly given by way of

$$\eta_l(\boldsymbol{x},t) = \begin{cases} \chi(\boldsymbol{x})s_l\left(\boldsymbol{\Xi}(\boldsymbol{x},t)\right), & l \in \{1,\ldots,N/2\}, \\ \chi(\boldsymbol{x})c_l\left(\boldsymbol{\Xi}(\boldsymbol{x},t)\right), & l \in \{N/2+1,\ldots,N\}, \\ \widetilde{\chi}(\boldsymbol{x}), & l = N+1, \\ \widetilde{\chi}_{\nearrow}(\boldsymbol{x}), & l = N+2, \\ \widetilde{\chi}_{\rightarrow}(\boldsymbol{x}), & l = N+3. \end{cases}$$

The yet unspecified objects K, χ, $\widetilde{\chi}$, $(\widetilde{\chi}_l)_{l\in\{1,\ldots,K\}}$, $\boldsymbol{\Xi}$, $T^\star$, and $(t_a^l, t_b^l, \tau_l)_{l\in\{1,\ldots,K\}}$ are chosen effectively in Section 3 and solely depend on the fixed control region ω. In particular, they are independent of the initial state, the target state, the accuracy parameter, and the control time. In this sense, noting that σ is a scaling constant and $\gamma_{N+1}, \gamma_{N+2}, \gamma_{N+3}$ are expressed in terms of $(\zeta_{\boldsymbol{\ell}}^s, \zeta_{\boldsymbol{\ell}}^c)_{\boldsymbol{\ell}\in\mathcal{K}}$ via (2.12), the N unknown parameters $(\zeta_{\boldsymbol{\ell}}^s, \zeta_{\boldsymbol{\ell}}^c)_{\boldsymbol{\ell}\in\mathcal{K}} \subset \mathrm{L}^2((0,1);\mathbb{R})$ appearing explicitly in (2.8)–(2.10) are the actual controls in the system (1.1).

Example 2.2. A key aspect of this article is the possibility to choose $N = 4$ in Theorem 1.1. To this end, let us take $\mathcal{K} := \{[1,0]^\top, [0,1]^\top\}$ and observe that

$$\{s_{\boldsymbol{\ell}}(x_1,x_2), c_{\boldsymbol{\ell}}(x_1,x_2)\}_{\boldsymbol{\ell}\in\mathcal{K}} = \{\sin(x_1), \cos(x_1), \sin(x_2), \cos(x_2)\}\,.$$

As a result, the force $\widehat{\eta}$ is of the form

$$\begin{aligned}\widehat{\eta}(\boldsymbol{x},t) = \chi(\boldsymbol{x})\Big[&\widetilde{\zeta}_1(t)\sin\left(\Phi(\boldsymbol{x},t)\right) + \widetilde{\zeta}_2(t)\sin\left(\Psi(\boldsymbol{x},t)\right)\\ &+ \widetilde{\zeta}_3(t)\cos\left(\Phi(\boldsymbol{x},t)\right) + \widetilde{\zeta}_4(t)\cos\left(\Psi(\boldsymbol{x},t)\right)\Big],\end{aligned}$$

where the parameters $(\widetilde{\zeta}_i)_{i\in\{1,\ldots,4\}}$ are given by

$$\widetilde{\zeta}_i(t) := \sum_{j=1}^{K} \mathbb{I}_{[t_a^j, t_b^j]}(t)\zeta_i(\tau_j(t)), \quad i \in \{1,\ldots,4\}$$

and $\boldsymbol{\Xi} = [\Phi, \Psi]^\top$ is the flow determined in Section 3.4.

2.3 Controllability of the vorticity formulation

We establish two controllability results for the vorticity formulation (2.1). Several key arguments are outsourced to Section 5. To start with, we let $k := r - 1$, with $r \geq 2$ from Theorem 1.1, and consider for any $\delta > 0$ the scaled velocity average contributions

$$\widetilde{\boldsymbol{\aleph}}_\delta(t) := \int_{\mathbb{T}^2} \left(\delta^{-1}\overline{\boldsymbol{y}}(\boldsymbol{x}, \delta^{-1}t) + \boldsymbol{U}(\delta^{-1}t)\right)\,\mathrm{d}\boldsymbol{x}. \tag{2.14}$$

In (2.14), the profile $\boldsymbol{U}$ is the one from (2.3) and connects the averages of the prescribed initial and target velocity fields. Moreover, the function $\overline{\boldsymbol{y}}$ will be explicitly constructed in Section 3.3 and corresponds to a convection strategy on the torus. Further, it is reminded that, for given control parameters $(\zeta_l)_{l\in\{1,\ldots,K\}} \subset \mathrm{L}^2((0,T);\mathbb{R})$, the force $\widetilde{\eta} = \widetilde{\eta}_{\zeta_1,\ldots,\zeta_N}$ is defined in (2.10).

The following theorem, which is proved in Section 5.3, states the approximate controllability of the vorticity formulation (2.1) in small time via large controls. The "smallness" of the terminal time and the "largeness" of the controls are hereby reciprocal.

Theorem 2.3 (*cf.* Theorem 5.6)**.** *For any $h \in \mathrm{L}^2((0,1);\mathrm{H}^{k-1})$ and $w_0, w_1 \in \mathrm{H}^{k+1}$, there exist control parameters $(\zeta_l)_{l\in\{1,\dots,N\}} \subset \mathrm{L}^2((0,1);\mathbb{R})$ such that the sequence of solutions $(w_\delta \in \mathcal{X}^k_\delta)_{\delta>0}$ to the respective vorticity problems*

$$\partial_t w_\delta - \nu\Delta w_\delta + \left(\boldsymbol{\Upsilon}(w_\delta, \widetilde{\boldsymbol{\aleph}}_\delta) \cdot \boldsymbol{\nabla}\right) w_\delta = h + \delta^{-1}\widetilde{\eta}_{\zeta_1,\dots,\zeta_N}(\cdot, \delta^{-1}\cdot), \quad w_\delta(\cdot, 0) = w_0 \tag{2.15}$$

satisfies

$$w_\delta(\cdot, \delta) \longrightarrow w_1 \text{ in } \mathrm{H}^k \text{ as } \delta \longrightarrow 0. \tag{2.16}$$

Moreover, the convergence in (2.16) is uniform with respect to w_0, w_1 from a bounded subset of H^{k+1} and h from a bounded subset of $\mathrm{L}^2((0,1);\mathrm{H}^{k-1})$.

The next theorem is concerned with the approximate controllability of (2.1) on the original time interval $[0, T_{\mathrm{ctrl}}]$, thereby constituting a vorticity version of Theorem 1.1.

Theorem 2.4. *Let $T_{\mathrm{ctrl}} > 0$, $h \in \mathrm{L}^2((0,T_{\mathrm{ctrl}});\mathrm{H}^k)$, $w_0, w_1 \in \mathrm{H}^k$, and $\varepsilon > 0$ be arbitrary. There exist $\sigma \geq T_{\mathrm{ctrl}}^{-1}$ and control parameters*

$$(\zeta_l)_{l\in\{1,\dots,N\}} \subset \mathrm{L}^2((0,1);\mathbb{R})$$

such that the unique solution $w \in \mathcal{X}^k_{T_{\mathrm{ctrl}}}$ to (2.1), where η is obtained from $(\zeta_l)_{l\in\{1,\dots,N\}}$ through (2.11)–(2.13), satisfies the terminal condition

$$\|w(\cdot, T_{\mathrm{ctrl}}) - w_1\|_k < \varepsilon. \tag{2.17}$$

Proof. The idea is to first let the uncontrolled solution to (2.1) evolve for a while, just to activate proper controls shortly before the terminal time is reached (*cf.* Figure 3). However, as we want η to be exactly of the form (2.8), the main difficulty consists now of selecting via Theorem 2.3 a value of $\delta > 0$ which allows switching on the control precisely at the time $T_{\mathrm{ctrl}} - \delta$ while reaching the target region at $t = T_{\mathrm{ctrl}}$. Throughout the proof, the initial velocity average is abbreviated by $\boldsymbol{U}_0 := \int_{\mathbb{T}^2} \boldsymbol{u}_0(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$.

Step 1. Determining a suitable $\delta > 0$. We begin by fixing an arbitrary vortex $\widetilde{w}_1 \in \mathrm{H}^{k+1}$ such that $\|w_1 - \widetilde{w}_1\|_k < \varepsilon/2$, which is always possible by density. Moreover, let $\widetilde{w}_0 := S_{T_{\mathrm{ctrl}}}(w_0, h, \boldsymbol{U}_0)_{|t=T_{\mathrm{ctrl}}}$ be the state that is reached when the control is inactive during the whole time interval $[0, T_{\mathrm{ctrl}}]$. Due to the assumption that $h \in \mathrm{L}^2((0,T_{\mathrm{ctrl}});\mathrm{H}^k)$, one can verify with the aid of Lemma 2.1 and known parabolic smoothing effects (*cf.* [11,23]) that $S_{T_{\mathrm{ctrl}}}(w_0, h, \boldsymbol{U}_0) \in \mathrm{C}^0((0,T_{\mathrm{ctrl}}];\mathrm{H}^{k+1})$, hence $\widetilde{w}_0 \in \mathrm{H}^{k+1}$. By applying Theorem 2.3 with the initial and target states $\widetilde{w}_0$ and $\widetilde{w}_1$ respectively, while defining $h(\cdot, T_{\mathrm{ctrl}} - s) = 0$ if $s > T_{\mathrm{ctrl}}$, one obtains a small number $\delta \in (0, T_{\mathrm{ctrl}})$ and control parameters $(\rho_l)_{l\in\{1,\dots,N\}} \subset \mathrm{L}^2((0,1);\mathbb{R})$ such that the function

$$\widetilde{w}_\delta := S_\delta\left(\widetilde{w}_0, h(\cdot, T_{\mathrm{ctrl}} - \cdot) + \delta^{-1}\widetilde{\eta}_{\rho_1,\dots,\rho_N}(\cdot, \delta^{-1}\cdot), \widetilde{\boldsymbol{\aleph}}_\delta\right) \tag{2.18}$$

meets the terminal condition

$$\|\widetilde{w}_\delta(\cdot,\delta) - \widetilde{w}_1\|_k < \varepsilon/2. \tag{2.19}$$

Then, since $S_{T_{\mathrm{ctrl}}}(w_0, h, \boldsymbol{U}_0) \in \mathrm{C}^0((0,T_{\mathrm{ctrl}}];\mathrm{H}^{k+1})$, and because Theorem 2.3 allows choosing δ uniformly with respect to initial data from a bounded subset of H^{k+1}, we can assume $\delta \in (0, T_{\mathrm{ctrl}})$ to be sufficiently small such that (2.19) remains valid when $\widetilde{w}_0$ is replaced by a different state from a bounded subset $B \subset \mathrm{H}^{k+1}$ with

$$\left\{S_{T_{\mathrm{ctrl}}}(w_0, h, \boldsymbol{U}_0)_{|_{t=s}} \mid s \in [T_{\mathrm{ctrl}} - \delta, T_{\mathrm{ctrl}}]\right\} \subset B.$$

Now, the value of δ is set and this step is complete. However, for the sake of clarity, let us make two additional remarks.

First remark. The function $\widetilde{w}_\delta$ together with the associated controls $(\rho_l)_{l\in\{1,\dots,N\}}$ will not be used in the sequel, as they were only introduced for the purpose of explaining the choice of $\delta > 0$. In particular, we only took in (2.18) the external force time reversed in order to build a controlled trajectory defined on a suitably short time interval $[0,\delta]$ without actually knowing δ beforehand.

Second remark. Since the convergence property (2.16) in Theorem 2.3 is uniform with respect to prescribed external forces from a bounded subset of $\mathrm{L}^2((0,1);\mathrm{H}^{k-1})$, by noticing that

$$\int_0^\delta \|h(\cdot, T_{\mathrm{ctrl}} - s)\|_{k-1}^2 \,\mathrm{d}s = \int_0^\delta \|h(\cdot, T_{\mathrm{ctrl}} - \delta + s)\|_{k-1}^2 \,\mathrm{d}s, \tag{2.20}$$

one can infer that also

$$\widehat{w}_\delta \coloneqq S_\delta\left(\widetilde{w}_0, h(\cdot, T_{\mathrm{ctrl}} - \delta + \cdot) + \delta^{-1}\widetilde{\eta}_{\rho_1,\dots,\rho_N}(\cdot, \delta^{-1}\cdot), \widetilde{\boldsymbol{\aleph}}_\delta\right) \tag{2.21}$$

satisfies the terminal condition

$$\|\widehat{w}_\delta(\cdot,\delta) - \widetilde{w}_1\|_k < \varepsilon/2. \tag{2.22}$$

The trajectory $\widehat{w}_\delta$ from (2.21) involves the correct external forcing, but is only of illustrative purpose here. Notably, since the forcing $h(\cdot, T_{\mathrm{ctrl}} - \delta + \cdot)$ depends on δ, we could not have resorted to Theorem 2.3 for finding δ such that $\widehat{w}_\delta$ defined via (2.21) obeys (2.22) in the first place, as this is a chicken or egg problem.

Step 2. Gluing two trajectories. By applying Theorem 2.3 with the initial state $w_{\delta,0} \coloneqq S_{T_{\mathrm{ctrl}}-\delta}(w_0, h, \boldsymbol{U}_0)|_{t=T_{\mathrm{ctrl}}-\delta}$ and the now known external force $h(\cdot, T_{\mathrm{ctrl}} - \delta + \cdot)$, we retrieve control parameters $(\zeta_l)_{l\in\{1,\dots,N\}} \subset \mathrm{L}^2((0,1);\mathbb{R})$ and a number $\varkappa_0 > 0$ such that the function

$$w_\varkappa = S_\varkappa\left(w_{\delta,0}, h(\cdot, T_{\mathrm{ctrl}} - \delta + \cdot) + \varkappa^{-1}\widetilde{\eta}_{\zeta_1,\dots,\zeta_N}(\cdot, \varkappa^{-1}\cdot), \widetilde{\boldsymbol{\aleph}}_\varkappa\right) \tag{2.23}$$

fulfills the terminal condition

$$\|w_\varkappa(\cdot,\varkappa) - \widetilde{w}_1\|_k < \varepsilon/2 \tag{2.24}$$

for any $\varkappa \in (0,\varkappa_0)$. Because the convergence property (2.16) in Theorem 2.3 is uniform with respect to initial data from a bounded subset of H^{k+1} and external forces from a bounded subset of $\mathrm{L}^2((0,1);\mathrm{H}^{k-1})$, keeping (2.20) in mind, we can take $\varkappa = \delta < \varkappa_0$ in (2.23).

As a result, the proof of Theorem 2.4 can now be concluded by choosing $\sigma := \delta^{-1}$ and defining

$$w(\boldsymbol{x},t) := \begin{cases} S_{T_{\mathrm{ctrl}}}(w_0, h, \boldsymbol{U}_0)(\boldsymbol{x},t) & \text{when } (\boldsymbol{x},t) \in \mathbb{T}^2 \times [0,T_\sigma], \\ w_{\sigma^{-1}}(\boldsymbol{x}, t - T_\sigma) & \text{when } (\boldsymbol{x},t) \in \mathbb{T}^2 \times [T_\sigma, T_{\mathrm{ctrl}}], \end{cases} \tag{2.25}$$

where $T_\sigma = T_{\mathrm{ctrl}} - \sigma^{-1}$. By Lemma 2.1, the vortex w from (2.25) constitutes the unique solution to (2.1) with the control $\eta = \eta_{\zeta_1,\dots,\zeta_N,\sigma}$ as specified in (2.8). Owing to (2.24) with $\varkappa = \sigma^{-1}$, the function w defined in (2.25) obeys (2.17). □

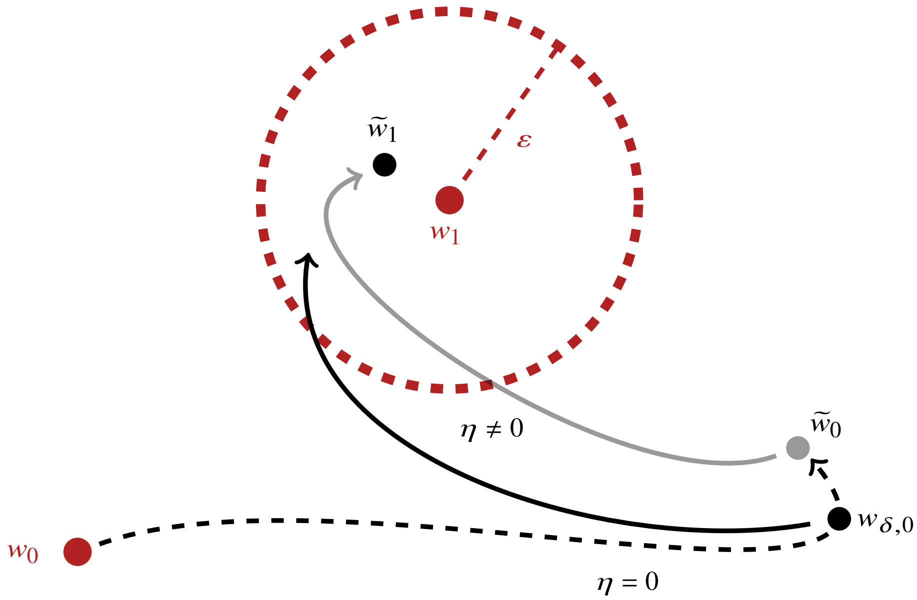


Figure 3: A sketch of several ideas from the proof of Theorem 2.4. Starting at $t = 0$ with the original initial state w_0, we first follow the uncontrolled trajectory (dashed line with arrow) until reaching a state $w_{\delta,0}$ that is very close to the terminal point $\widetilde{w}_0$ of the uncontrolled trajectory. Hereto, we determine a good $\delta > 0$ for steering the system from the state $\widetilde{w}_0$ to a small neighborhood of the regularized target vorticity $\widetilde{w}_1$. After replacing the starting point of the controlled trajectory by $w_{\delta,0}$, we can still employ the same δ for reaching a small neighborhood of $\widetilde{w}_1$ with the help of different controls.

2.4 Conclusion of the main theorem

We can now conclude Theorem 1.1 as a corollary of Theorem 2.4. To this end, by resorting to the elliptic theory for the div-curl problem (2.4), we fix any constant $C_0 > 0$ such that

$$\|\boldsymbol{\Upsilon}(z, \boldsymbol{A})\|_{k+1} \leq C_0(\|z\|_k + |\boldsymbol{A}|) \tag{2.26}$$

for all $z \in \mathrm{H}^k$ and $\boldsymbol{A} \in \mathbb{R}^2$. Moreover, we denote by $\varepsilon > 0$ the approximation accuracy parameter selected in Theorem 1.1.

Step 1. Determining the control parameters. To begin with, we apply Theorem 2.4 with the data chosen as

$$w_0 \coloneqq \boldsymbol{\nabla} \wedge \boldsymbol{u}_0, \quad w_1 \coloneqq \boldsymbol{\nabla} \wedge \boldsymbol{u}_1, \quad h \coloneqq \boldsymbol{\nabla} \wedge \boldsymbol{f}.$$

This provides $\sigma_0 \geq T_{\mathrm{ctrl}}^{-1}$ and the existence of controls $\zeta_1, \dots \zeta_N \in \mathrm{L}^2((0, T_{\mathrm{ctrl}}); \mathbb{R})$ such that the unique solution w to (2.1), with $\eta = \eta_{\zeta_1, \dots \zeta_N, \sigma}$ given by (2.8), satisfies

$$\|w(\cdot, T_{\mathrm{ctrl}}) - w_1\|_k < \frac{\varepsilon}{C_0}, \tag{2.27}$$

for any $\sigma \geq \sigma_0$ and where $C_0 > 0$ is the constant determined by (2.26).

Step 2. Integrating the vorticity controls. Anticipating the definitions of the cutoffs χ, $\widetilde{\chi}$, and $(\widetilde{\chi}_l)_{l \in \{1,\dots,K\}}$ from Section 3.2, which are used in the description of η given in Section 2.2, we can choose a point $\boldsymbol{p}^{\omega} = [p_1^{\omega}, p_2^{\omega}]^{\top} \in \mathbb{T}^2$, a small number $d^{\omega} > 0$, and a length parameter $L^{\omega} > 0$ such that the square

$$\mathcal{O}^{\omega} \coloneqq \boldsymbol{p}^{\omega} + [0, L^{\omega}]^2$$

satisfies

$$\operatorname{supp}(\eta) \subset \mathcal{O}^{\omega} \subset \omega, \quad \operatorname{dist}(\mathcal{O}^{\omega}, \partial\omega) > d^{\omega},$$

where $\partial\omega$ is the boundary of the control region $\omega \subset \mathbb{T}^2$. Therefore, inspired by [10, Appendix A.2], we denote the auxiliary functions

$$a(x_1, x_2, t) \coloneqq \int_{p_1^{\omega}}^{x_1} \eta(s, x_2, t) \, \mathrm{d}s, \quad b(x_2, t) \coloneqq \int_{p_2^{\omega}}^{x_2} a(p_1^{\omega} + L^{\omega}, s, t) \, \mathrm{d}s.$$

Moreover, we select a profile $\rho \in \mathrm{C}^{\infty}(\mathbb{T}^1; \mathbb{R})$ which satisfies

$$\operatorname{supp}(\rho) \subset (p_1^{\omega}, p_1^{\omega} + L^{\omega} + d^{\omega}), \quad \rho(s) = 1 \text{ for } s \in (p_1^{\omega} + L^{\omega}, p_1^{\omega} + L^{\omega} + d^{\omega}/2),$$

where $\mathbb{T}^1 := \mathbb{R}/2\pi\mathbb{Z}$ stands for the one-dimensional torus. Another auxiliary profile is then given via

$$c(x_1, x_2, t) := a(x_1, x_2, t) - \rho(x_1) a(p_1^{\omega} + L^{\omega}, x_2, t).$$

Piecing together these building blocks, a vector field $\boldsymbol{\xi} = [\xi_1, \xi_2]^\top$ is given by way of assigning to all $(x_1, x_2, t) \in \mathbb{T}^2 \times (0, T_{\mathrm{ctrl}})$ the values

$$\xi_1(x_1, x_2, t) := \begin{cases} -\overline{\chi}(x_1, x_2) \frac{\mathrm{d}\rho}{\mathrm{d}s}(x_1) b(x_2, t), & x_1 \in [0, p_1^{\omega} + L^{\omega} + d^{\omega}/4), \\ 0, & x_1 \in [p_1^{\omega} + L^{\omega} + d^{\omega}/4, 2\pi) \end{cases} \tag{2.28}$$

and

$$\xi_2(x_1, x_2, t) := \begin{cases} 0, & x_1 \in [0, p_1^{\omega}), \\ \overline{\chi}(x_1, x_2) c(x_1, x_2, t), & x_1 \in [p_1^{\omega}, p_1^{\omega} + L^{\omega} + d^{\omega}/4], \\ 0, & x_1 \in (p_1^{\omega} + L^{\omega} + d^{\omega}/4, 2\pi), \end{cases} \tag{2.29}$$

where the phantom cutoff $\overline{\chi} \in \mathrm{C}^{\infty}(\mathbb{T}^2; [0,1])$ obeys $\overline{\chi}\boldsymbol{\xi} = \boldsymbol{\xi}$ and is selected such that the proclaimed profiles $\boldsymbol{\vartheta}_1, \dots, \boldsymbol{\vartheta}_{N+3}$ in (1.4) will be supported in ω, which, for instance, is achieved by requiring that

$$\boldsymbol{p}^{\omega} + [0, L^{\omega} + d^{\omega}/2]^2 \subset \{\overline{\chi} = 1\} \subset \operatorname{supp}(\overline{\chi}) \subset \omega.$$

The force $\boldsymbol{\xi} = \boldsymbol{\xi}_{\zeta_1, \dots, \zeta_N, \sigma}$, defined through (2.28) and (2.29), enjoys (at least) the same regularity as $\eta_{\zeta_1, \dots, \zeta_N, \sigma}$, thus $\boldsymbol{\xi} \in \mathrm{L}^2((0, T_{\mathrm{ctrl}}); \mathrm{C}^{\infty}(\mathbb{T}^2; \mathbb{R}^2))$. In addition, by noting that $\int_{\mathbb{T}^2} \eta(\boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x} = 0$ and $\operatorname{supp}(\eta(\cdot, t)) \subset \omega$ hold for all $t \in [0, T_{\mathrm{ctrl}}]$, one can infer the properties

$$\operatorname{supp}(\boldsymbol{\xi}(\cdot, t)) \subset \omega, \quad \nabla \wedge (\boldsymbol{\xi}(\cdot, t)) = \eta(\cdot, t).$$

Finally, after obtaining the explicit profiles $\boldsymbol{\vartheta}_1, \dots, \boldsymbol{\vartheta}_{N+3}$ by inserting the expression (2.8) into (2.28) and (2.29), one arrives at the representation

$$\boldsymbol{\xi}(\boldsymbol{x}, t) = \sum_{l=1}^{N} \gamma_l(t) \boldsymbol{\vartheta}_l(\boldsymbol{x}, 1 - \sigma(T_{\mathrm{ctrl}} - t)) + \sum_{l=N+1}^{N+3} \gamma_l(t) \boldsymbol{\vartheta}_l(\boldsymbol{x}) + \widetilde{\boldsymbol{\xi}}(\boldsymbol{x}, t), \tag{2.30}$$

where the coefficients $(\gamma_j)_{j \in \{1, \dots, N\}}$ are those from (2.11), and it solely remains to determine the part of $\boldsymbol{\xi}$ acting on the velocity average in (1.1), namely

$$\widetilde{\boldsymbol{\xi}}(\boldsymbol{x}, t) := \gamma_{N+4}(t) \boldsymbol{\vartheta}_{N+4} + \gamma_{N+5}(t) \boldsymbol{\vartheta}_{N+5}.$$

Up to composing $\boldsymbol{\vartheta}_l(\boldsymbol{x}, \cdot)$ for $l \in \{1, \dots, N\}$ with the transformation $t \mapsto 1 - t$, one can also write (2.30) in the form (1.3), accepting the abuse of notation

$$\boldsymbol{\vartheta}_l(\boldsymbol{x}, 1 - \sigma(T_{\mathrm{ctrl}} - t)) = \boldsymbol{\vartheta}_l(\boldsymbol{x}, \sigma(T_{\mathrm{ctrl}} - t)).$$

Step 3. Choosing $\boldsymbol{\vartheta}_{N+4}, \boldsymbol{\vartheta}_{N+5}$. We select $\boldsymbol{\vartheta}_{N+4}, \boldsymbol{\vartheta}_{N+5} \in \mathrm{C}^{\infty}(\mathbb{T}^2;\mathbb{R}^2)$ in an explicit way only depending on Ω. Subsequently, the controls $\gamma_{N+4}, \gamma_{N+5} \in \mathrm{L}^2((0,T_{\mathrm{ctrl}});\mathbb{R})$ are determined. To this purpose, let us fix two cutoff vector fields $\boldsymbol{\Lambda}, \boldsymbol{\Sigma} \in \mathrm{C}^{\infty}(\mathbb{T}^2;\mathbb{R}^2)$ satisfying $\mathbb{R}^2 = \operatorname{span}_{\mathbb{R}}\{\int_{\mathbb{T}^2} \boldsymbol{\Lambda}\,\mathrm{d}\boldsymbol{x}, \int_{\mathbb{T}^2} \boldsymbol{\Sigma}\,\mathrm{d}\boldsymbol{x}\}$ and

$$\boldsymbol{\nabla} \wedge \boldsymbol{\Lambda} = \boldsymbol{\nabla} \wedge \boldsymbol{\Sigma} = 0, \quad \operatorname{supp}(\boldsymbol{\Lambda}) \cup \operatorname{supp}(\boldsymbol{\Sigma}) \subset \Omega. \tag{2.31}$$

This choice is possible due to the assumption that Ω contains two curves $C_1, C_2 \subset \Omega$ with the property that $\mathbb{T}^2 \setminus (C_1 \cup C_2)$ is simply-connected (*cf.* Figure 1). For the sake of simplicity, let us for the moment being assume that C_1 and C_2 can be chosen as the graphs of smooth functions over the vertical and horizontal axis respectively, noting that the arguments for the general case are provided in Theorem A.1. Therefore, one can explicitly construct two functions $\upsilon_1, \upsilon_2 \colon \mathbb{T}^1 \longrightarrow \mathbb{R}$ such that

$$x_1 + \upsilon_1(x_2) = 0 \iff (x_1,x_2) \in C_1, \quad x_2 + \upsilon_2(x_1) = 0 \iff (x_1,x_2) \in C_2.$$

Furthermore, we fix a small number $l > 0$ and a cutoff $\beta \in \mathrm{C}^{\infty}(\mathbb{T}^1;\mathbb{R}_+)$ with

$$\forall i \in \{1,2\}\colon \ \operatorname{dist}(C_i, \partial\Omega) < l, \quad \operatorname{supp}(\beta) \subset (-l/2, l/2), \quad \beta(0) > 0.$$

On this groundwork, we then define the vector fields

$$\boldsymbol{\Lambda} \coloneqq \begin{bmatrix} \beta(x_1 + \upsilon_1(x_2)) \\ \beta(x_1 + \upsilon_1(x_2))\frac{\mathrm{d}\upsilon_1}{\mathrm{d}s}(x_2) \end{bmatrix}, \quad \boldsymbol{\Sigma} \coloneqq \begin{bmatrix} \beta(x_2 + \upsilon_2(x_1))\frac{\mathrm{d}\upsilon_2}{\mathrm{d}s}(x_1) \\ \beta(x_2 + \upsilon_2(x_1)) \end{bmatrix}, \tag{2.32}$$

which obey (2.31) and have linearly independent averages, followed by assigning the new names

$$\boldsymbol{\vartheta}_{N+4} \coloneqq \boldsymbol{\Lambda}, \quad \boldsymbol{\vartheta}_{N+5} \coloneqq \boldsymbol{\Sigma}.$$

Now, we can choose the controls γ_{N+4} and γ_{N+5}, depending on all the data of the controllability problem, by way of

$$\gamma_{N+4}(t) = \partial_t A_1(t) - B_1(t), \quad \gamma_{N+5}(t) = \partial_t A_2(t) - B_2(t), \tag{2.33}$$

where the coordinates A_1, A_2, B_1, B_2 are uniquely determined such that

$$\aleph_{\sigma}(t) = A_1(t)\int_{\mathbb{T}^2} \boldsymbol{\Lambda}\,\mathrm{d}\boldsymbol{x} + A_2(t)\int_{\mathbb{T}^2} \boldsymbol{\Sigma}\,\mathrm{d}\boldsymbol{x},$$

$$\int_{\mathbb{T}^2} \left(\widehat{\boldsymbol{\xi}}(\boldsymbol{x},t) + \boldsymbol{f}(\boldsymbol{x},t)\right)\mathrm{d}\boldsymbol{x} = B_1(t)\int_{\mathbb{T}^2} \boldsymbol{\Lambda}\,\mathrm{d}\boldsymbol{x} + B_2(t)\int_{\mathbb{T}^2} \boldsymbol{\Sigma}\,\mathrm{d}\boldsymbol{x},$$

with $\aleph_{\sigma}(t)$ from (2.3) and $\widehat{\boldsymbol{\xi}}$ given by

$$\widehat{\boldsymbol{\xi}}(\boldsymbol{x},t) \coloneqq \sum_{l=1}^{N} \gamma_l(t)\boldsymbol{\vartheta}_l(\boldsymbol{x}, 1 - \sigma(T_{\mathrm{ctrl}} - t)) + \sum_{l=N+1}^{N+3} \gamma_l(t)\boldsymbol{\vartheta}_l(\boldsymbol{x}).$$

In light of the formulas (2.11)–(2.13) and the previously determined profiles $\boldsymbol{\vartheta}_1, \ldots, \boldsymbol{\vartheta}_{N+3}$, the definitions in (2.33) allow to describe γ_{N+4} and γ_{N+5} explicitly in terms of the controls $\gamma_1, \ldots, \gamma_N$.

Step 4. Conclusion. Let $\boldsymbol{u} \in \mathrm{C}^0([0, T_{\mathrm{ctrl}}]; \mathbf{V}^{k+1}) \cap \mathrm{L}^2((0, T_{\mathrm{ctrl}}); \mathbf{V}^{k+2})$ be the unique solution to (1.1) associated with the control force $\boldsymbol{\xi} = \boldsymbol{\xi}_{\gamma_1,\ldots,\gamma_N,\sigma}$. We note that, by integrating the velocity equation of the Navier–Stokes problem (1.1), one has

$$\int_{\mathbb{T}^2} \boldsymbol{u}(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{T}^2} \boldsymbol{u}(\boldsymbol{x}, 0)\, \mathrm{d}\boldsymbol{x} + \int_0^t \left(\int_{\mathbb{T}^2} \boldsymbol{\xi}(\boldsymbol{x}, s)\, \mathrm{d}\boldsymbol{x} + \int_{\mathbb{T}^2} \boldsymbol{f}(\boldsymbol{x}, s)\, \mathrm{d}\boldsymbol{x} \right) \mathrm{d}s.$$

As a result, by taking the estimates (2.26) and (2.27) into account, $\boldsymbol{u}$ is seen to satisfy the terminal condition

$$\|\boldsymbol{u}(\cdot, T_{\mathrm{ctrl}}) - \boldsymbol{u}_1\|_{k+1} \leq C_0 \|w(\cdot, T_{\mathrm{ctrl}}) - w_1\|_k < \varepsilon,$$

which completes the proof of Theorem 1.1.

3 Effective construction of the control force

The yet unspecified objects K, χ, $\widetilde{\chi}$, $(\widetilde{\chi}_i)_{i\in\{1,\ldots,K\}}$, $\boldsymbol{\Xi}$, $(t_a^l, t_b^l, \tau_l)_{l\in\{1,\ldots,K\}}$, $T^\star$, which already appeared in (2.8)–(2.10), are introduced in this section. The motivations for the choices made below will become apparent in the proof of Theorem 5.3.

3.1 Open covering by overlapping squares

Let us take any small number $d > 0$ such that $\omega_d := \{\boldsymbol{x} \in \omega \mid \operatorname{dist}(\boldsymbol{x}, \partial\omega) > d\}$ is non-empty. Moreover, we choose length and height parameters

$$0 < L_1 < L_2 < 2\pi, \quad 0 < H_1 < H_2 < 2\pi$$

in a way that $[L_1, L_2] \times [H_1, H_2] \subset \omega_d$. Subsequently, we fix a possibly large square number $K \in \mathbb{N}$ which satisfies $K > 1$ and

$$l_K := \frac{2\pi}{\sqrt{K} - 1} < \frac{1}{3} \min\{L_2 - L_1, H_2 - H_1\}.$$

By employing the above notations, the torus $\mathbb{T}^2$ may be covered by overlapping squares $(O_i)_{i\in\{1,\ldots,K\}}$ (*cf.* Figure 4), each being a rigid translation of $(0, l_K)^2$. For the sake of explicitness, and in order to fix their enumeration, we assume that the bottom left corners of $O_1, \ldots, O_K$ are given by the respective points

$$\boldsymbol{x}_1 = [x_{1,1}, x_{1,2}]^\top, \ldots, \boldsymbol{x}_K = [x_{K,1}, x_{K,2}]^\top \in \mathbb{T}^2,$$

defined via

$$x_{i+\sqrt{K}(l-1),1} = \frac{2\pi(i-1)}{\sqrt{K}}, \quad x_{i+\sqrt{K}(l-1),2} = \frac{2\pi(l-1)}{\sqrt{K}}, \quad i,l = 1,\dots,\sqrt{K}.$$

Lastly, let a reference square O, with its bottom left corner denoted by $\boldsymbol{p}_K$, be placed inside the rectangle $[L_1, L_2] \times [H_1, H_2]$ via

$$\boldsymbol{p}_K := \left(\frac{L_1 + L_2 - l_K}{2}, \frac{H_1 + H_2 - l_K}{2}\right), \quad O := \boldsymbol{p}_K + (0, l_K)^2.$$

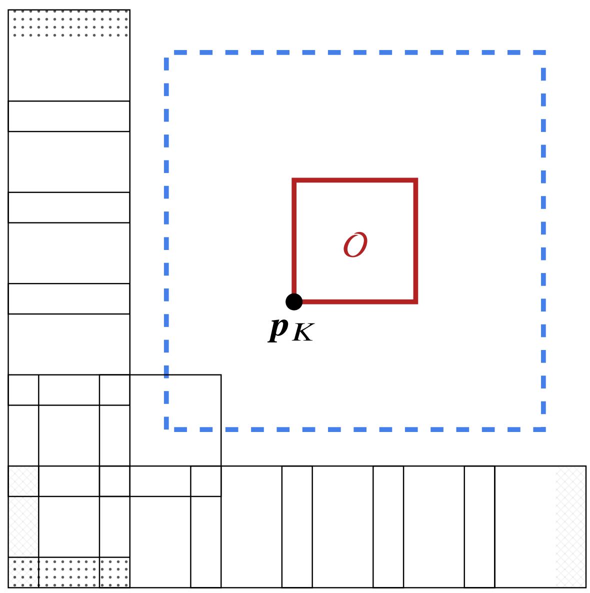


Figure 4: An illustration of the chosen covering for $\mathbb{T}^2$ by $K = 36$ overlapping squares. Areas which overlap due to periodicity are filled with a corresponding pattern. Only a few squares are depicted. The (red) reference square O is located in the interior of the control region and $\boldsymbol{p}_K$ denotes its bottom left corner. The (blue) dashed rectangle represents $[L_1, L_2] \times [H_1, H_2]$.

3.2 Partition of unity

We introduce a special partition of unity with respect to $(O_l)_{l\in\{1,\dots,K\}}$ arising from rigid translations of a single cutoff function. To begin with, let $\widetilde{\mu} \in \mathrm{C}^\infty(\mathbb{T}^1; [0,1])$ have the attributes (*cf.* Example 3.1)

$$\operatorname{supp}(\widetilde{\mu}) \subset (0, l_K), \quad \forall x \in \mathbb{T}^1 \colon \sum_{l=1}^{\sqrt{K}} \widetilde{\mu}\left(x + \frac{2\pi(l-1)}{\sqrt{K}}\right) = 1 \tag{3.1}$$

and

$$\widetilde{\mu}(s) = 1 \iff s \in \left[\frac{2\pi}{K - \sqrt{K}}, \frac{2\pi}{\sqrt{K}}\right]. \tag{3.2}$$

Thereafter, we define the cutoff functions $\mu, \chi \in \mathrm{C}^\infty(\mathbb{T}^2; [0,1])$ by virtue of

$$\mu(\boldsymbol{x}) \coloneqq \widetilde{\mu}(x_1)\widetilde{\mu}(x_2), \quad \chi(\boldsymbol{x}) \coloneqq \mu(\boldsymbol{x} - \boldsymbol{p}_K), \quad \boldsymbol{x} = [x_1, x_2]^\top \in \mathbb{T}^2. \tag{3.3}$$

As a result, a partition of unity with respect to the open covering $(\mathcal{O}_l)_{l\in\{1,\dots,K\}}$ is given by the family of translations

$$(\mu_l \coloneqq \mu(\cdot - \boldsymbol{x}_l))_{l=1,\dots,K} \subset \mathrm{C}^\infty(\mathbb{T}^2; [0,1]). \tag{3.4}$$

In particular, for all $l \in \{1, \dots, K\}$ one has $\operatorname{supp}(\mu_l) \subset \mathcal{O}_l$ while $\{\operatorname{supp}(\mu_l) = 1\}$ describes the region of $\mathcal{O}_l$ which does not overlap with $\mathcal{O}_i$ for $i \neq l$, and it holds

$$\forall \boldsymbol{x} \in \mathbb{T}^2 \colon \sum_{l=1}^{K} \mu_l(\boldsymbol{x}) = 1.$$

Finally, any smooth cutoff function $\widetilde{\chi} \in \mathrm{C}^\infty(\mathbb{T}^2; \mathbb{R}_+)$ with $\int_{\mathbb{T}^2} \widetilde{\chi}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 1$ is selected such that $\operatorname{supp}(\widetilde{\chi}) \subset \{\chi = 1\}$, followed by assigning to each $j \in \{1, \dots, K\}$ one of the two shifted versions via

$$\widetilde{\chi}_j \coloneqq \begin{cases} \widetilde{\chi}_{\nearrow} \coloneqq \widetilde{\chi}(\cdot - \frac{2\pi}{\sqrt{K}}\boldsymbol{e}_1 - \frac{2\pi}{\sqrt{K}}\boldsymbol{e}_2) & \text{if } j \text{ is a multiple of } \sqrt{K}, \\ \widetilde{\chi}_{\rightarrow} \coloneqq \widetilde{\chi}(\cdot - \frac{2\pi}{\sqrt{K}}\boldsymbol{e}_1) & \text{otherwise.} \end{cases} \tag{3.5}$$

Example 3.1. Consider any $\widehat{\mu} \in \mathrm{C}_0^\infty((-l_K, l_K); [0,1])$ obeying $\widehat{\mu}(s) = 0$ if and only if $s \in [-l_K, l_K] \backslash (0, l_K - 1/2K)$ and $\widehat{\mu}(s) = 1$ if and only if $s \in [2\pi/(K - \sqrt{K}), l_K - 1/K]$. Subsequently, a reference cutoff $\widetilde{\mu} \in \mathrm{C}^\infty(\mathbb{T}^1; [0,1])$ with the properties (3.1) and (3.2) is given by

$$\widetilde{\mu}(s) = \mathbb{I}_{\left[0, \frac{2\pi}{K-\sqrt{K}}\right]}(x)\widehat{\mu}(x) + \mathbb{I}_{\left(\frac{2\pi}{K-\sqrt{K}}, \frac{2\pi}{\sqrt{K}}\right)}(x) + \mathbb{I}_{\left[\frac{2\pi}{\sqrt{K}}, l_K\right]}(x)\left(1 - \widehat{\mu}\left(x - \frac{2\pi}{\sqrt{K}}\right)\right).$$

3.3 Convection strategy on the torus

Inspired by classical applications of the return method to fluid problems (*cf.* [9, Part 2, Chapter 6.2]), we build a vector field along which information originating anywhere on the torus will eventually pass through the control region. This vector field has to be of a very specific nature, while being obtained in a constructive way. To start with, the reference time interval $[0,1]$ is subdivided by the points

$$0 < T_a = t_c^0 < t_a^1 < t_b^1 < t_c^1 < t_a^2 < t_b^2 < t_c^2 < \cdots < t_a^K < t_b^K < t_c^K = T_b < 1,$$

which are, for simplicity, chosen to be of equal distance denoted by $T^\star > 0$, that is

$$t_a^l - t_c^{l-1} = t_c^l - t_b^l = t_b^l - t_a^l = T_a = 1 - T_b = T^\star, \quad l \in \{1, \dots, K\}.$$

Moreover, the construction of localized controls in Section 5.2 will involve the functions of time

$$\tau_l(t) := \mathbb{I}_{[t_a^l, t_b^l]}(t) \left(T_b + t - t_a^l\right), \quad \overline{\tau}(t) := \sum_{l=1}^{K} \tau_l(t). \tag{3.6}$$

Hereafter, a smooth profile $\overline{\boldsymbol{y}} \in \mathrm{C}_0^\infty((0,1);\mathbb{T}^2)$, which can at each time be interpreted as a constant vector field, is constructed such that $\overline{\boldsymbol{y}}$ together with its flow $\boldsymbol{\mathcal{Y}}$, obtained by solving the system of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\mathcal{Y}}(\boldsymbol{x}, s, t) = \overline{\boldsymbol{y}}(\boldsymbol{\mathcal{Y}}(\boldsymbol{x}, s, t), t), \quad \boldsymbol{\mathcal{Y}}(\boldsymbol{x}, s, s) = \boldsymbol{x}, \quad s, t \in [0, 1], \tag{3.7}$$

possess the following properties:

P1) $\overline{\boldsymbol{y}}(t) = \boldsymbol{0}$ for all $t \in [0, T_a] \cup [T_b, 1]$;

P2) $\boldsymbol{\mathcal{Y}}(\boldsymbol{x}, 0, 1) = \boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{T}^2$;

P3) each square O_l is transported by $\boldsymbol{\mathcal{Y}}$ into the reference square $O \subset \omega$ and pauses there during the time interval $[t_a^l, t_b^l]$, that is

$$\forall l \in \{1, \ldots, K\},\, \forall t \in [t_a^l, t_b^l]\colon\ \boldsymbol{\mathcal{Y}}(O_l, 0, t) = O.$$

Remark 3.2. As a consequence of the fact that $\overline{\boldsymbol{y}}$ does not depend on $\boldsymbol{x}$, the flow $\boldsymbol{\mathcal{Y}}$ rigidly translates $\mathbb{T}^2$ as a whole. Therefore, the property P3 implies

$$\forall l \in \{1, \ldots, K\},\, \forall \boldsymbol{x} \in O_l,\, \forall t \in [t_a^l, t_b^l]\colon\ \boldsymbol{\mathcal{Y}}(\boldsymbol{x}, 0, t) = \boldsymbol{p}_K + (\boldsymbol{x} - \boldsymbol{x}_l).$$

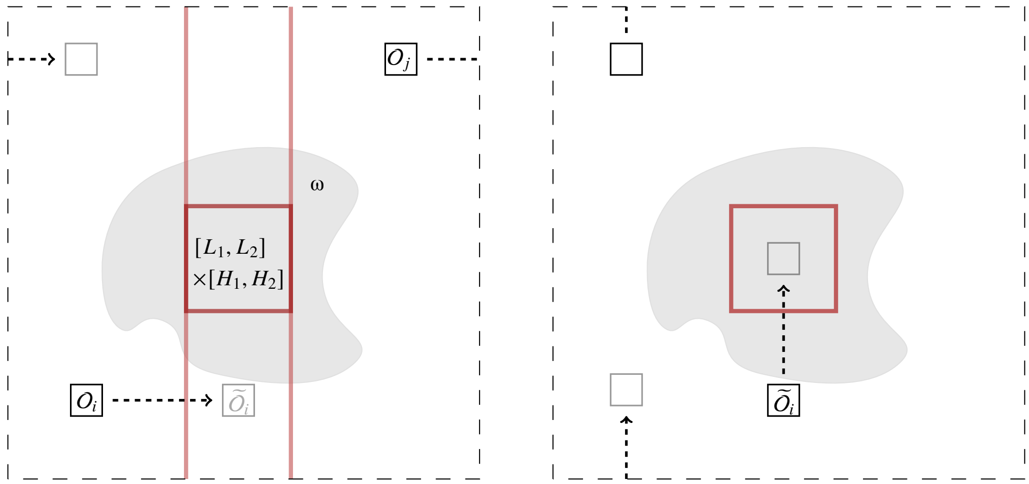
Theorem 3.3. *There exists $\overline{\boldsymbol{y}} \in \mathrm{C}_0^\infty((0,1);\mathbb{T}^2)$ satisfying the properties* P1–P3.

Proof. The proof is based on performing a sequence of horizontal and vertical translations of the whole torus $\mathbb{T}^2$, as illustrated in Figure 5. One could also use other translations, for instance, shifts in the direction of $\boldsymbol{p}_K - \boldsymbol{x}_l$ for each $l \in \{1, \ldots, K\}$ respectively. To begin with, for each $i \in \{1, \ldots, K\}$ let $h_i \in \mathrm{C}_0^\infty((0, T^\star/2);\mathbb{R})$ be any function which obeys

$$x_{i,1} + \int_0^{T^\star/2} h_i(s)\,\mathrm{d}s = \frac{L_1 + L_2 - l_K}{2}.$$

Similarly, for each $i \in \{1, \ldots, K\}$ take $v_i \in \mathrm{C}_0^\infty((0, T^\star/2);\mathbb{R})$ such that

$$x_{i,2} + \int_0^{T^\star/2} v_i(s)\,\mathrm{d}s = \frac{H_1 + H_2 - l_K}{2}.$$



(a) Shifting the whole torus $\mathbb{T}^2$ to the right such that the points from the square O_i are moved to $\widetilde{O}_i$, which is situated in the horizontal center of $[L_1, L_2] \times [0, 2\pi]$.

(b) Shifting the whole torus $\mathbb{T}^2$ upwards in a way that the points originating from the square $\widetilde{O}_i$ are moved to the center of $[L_1, L_2] \times [H_1, H_2]$.

Figure 5: The vector function $\overline{\boldsymbol{y}}$, which only depends on time, is constructed by a sequence of horizontal and vertical shifts of the whole torus $\mathbb{T}^2$. In order to render the constructions smooth, there is a scheduled pause between each shift. The gray region in the background depicts the set ω containing the support of the vorticity controls.

Based on this groundwork, we introduce the functions $\widetilde{\boldsymbol{\Theta}}_i \in \mathrm{C}_0^\infty((0,T^\star);\mathbb{R}^2)$ and $\widehat{\boldsymbol{\Theta}}_i \in \mathrm{C}_0^\infty((0,3T^\star);\mathbb{R}^2)$ via

$$\widetilde{\boldsymbol{\Theta}}_i(t) := \begin{cases} h_i(t)\boldsymbol{e}_1 & \text{if } t \in [0, T^\star/2], \\ v_i(t - T^\star/2)\boldsymbol{e}_2 & \text{if } t \in (T^\star/2, T^\star], \end{cases}$$

and respectively

$$\widehat{\boldsymbol{\Theta}}_i(t) := \begin{cases} \widetilde{\boldsymbol{\Theta}}_i(t) & \text{if } t \in [0, T^\star], \\ \mathbf{0} & \text{if } t \in [T^\star, 2T^\star], \\ -\widetilde{\boldsymbol{\Theta}}_i(t - 2T^\star) & \text{if } t \in (2T^\star, 3T^\star]. \end{cases}$$

Finally, a spatially constant vector field $\overline{\boldsymbol{y}} \in \mathrm{C}_0^\infty((0,1);\mathbb{T}^2)$, which satisfies the properties P1–P3, is defined as

$$\overline{\boldsymbol{y}}(t) := \begin{cases} \mathbf{0} & \text{if } t \in [0, T_a), \\ \widehat{\boldsymbol{\Theta}}_i(t - (3i-2)T^\star), & \text{if } t \in [(3i-2)T^\star, (3i+1)T^\star], \\ \mathbf{0} & \text{if } t \in (T_b, 1). \end{cases}$$

□

3.4 Flow maps obtained from observable families

It remains to effectively construct the flow map Ξ, which enters the definition of η in (2.8) through the auxilliary control $\widehat{\eta}$ described in (2.9). Hereto, we first select via Theorem 3.3 a profile $\overline{\boldsymbol{y}} \in \mathrm{C}_0^\infty((0,1);\mathbb{R}^2)$, together with the corresponding flow $\boldsymbol{\mathcal{Y}}$ defined as the solution to the system (3.7). We then proceed by recalling the notion of observability which has been introduced for the first time in [13] to study ergodicity properties of randomly forced partial differential equations; see also [17] where this concept is utilized for obtaining finite-dimensional controls that are supported in the whole torus.

Definition 3.4. Let $T > 0$ and $n \in \mathbb{N}$ be fixed. A family $(\phi_j)_{j\in\{1,\dots,n\}} \subset \mathrm{L}^2((0,T);\mathbb{R})$ is called observable if, for each subinterval $\mathrm{J} \subset (0,T)$, for any continuous function $b \in \mathrm{C}^0(\mathrm{J};\mathbb{R})$, and for all differentiable functions $(a_j)_{j\in\{1,\dots,n\}} \subset \mathrm{C}^1(\mathrm{J};\mathbb{R})$, one has the implication

$$b + \sum_{j=1}^{n} a_j\phi_j = 0 \text{ in } \mathrm{L}^2(\mathrm{J};\mathbb{R}) \quad \Longrightarrow \quad \forall t \in \mathrm{J}\colon b(t) = a_1(t) = \cdots = a_n(t) = 0.$$

Remark 3.5. As explained in [17, Section 3.3], see also [13], one can build observable families in the sense of Definition 3.4 in an explicit way and express them by closed formulas. For the sake of completeness, we briefly recall such a construction. Let $(\phi_j)_{j\in\{1,\dots,n\}}$ be any family of bounded measurable functions $(0,T) \longrightarrow \mathbb{R}$ such that each ϕ_j has well-defined left and right limits in the whole interval $(0,T)$. Next, take an arbitrary collection $(\mathrm{D}_i)_{i\in\{1,\dots,n\}} \subset (0,T)$ of disjoint countable sets which are dense in $(0,T)$, and which are chosen such that ϕ_j is discontinuous on D_j while, at the same time, being continuous on $(0,T)\setminus \mathrm{D}_j$. Then, one can show that $(\phi_j)_{j\in\{1,\dots,n\}}$ constitutes an observable family.

Given the finite set $\mathcal{K} \subset \mathbb{Z}^2_*$ selected in Section 2.2, we fix now any observable family $(\phi^{\mathrm{s}}_{\boldsymbol{\ell}}, \phi^{\mathrm{c}}_{\boldsymbol{\ell}})_{\boldsymbol{\ell}\in\mathcal{K}} \subset \mathrm{L}^2((0,T^\star);\mathbb{R})$ in the sense of Definition 3.4 and arbitrarily choose a function $\phi \in \mathrm{C}^1([0,T^\star];\mathbb{R})$ with the property

$$\phi(t) = 0 \iff t = T^\star.$$

On top of that, the family $(\psi^{\mathrm{s}}_{\boldsymbol{\ell}}, \psi^{\mathrm{c}}_{\boldsymbol{\ell}})_{\boldsymbol{\ell}\in\mathcal{K}} \subset \mathrm{W}^{1,2}((0,T^\star);\mathbb{R})$ of coefficients is taken as

$$\psi^{\mathrm{s}}_{\boldsymbol{\ell}}(t) := \phi(t)\int_0^t \phi^{\mathrm{s}}_{\boldsymbol{\ell}}(\sigma)\,\mathrm{d}\sigma, \quad \psi^{\mathrm{c}}_{\boldsymbol{\ell}}(t) := \phi(t)\int_0^t \phi^{\mathrm{c}}_{\boldsymbol{\ell}}(\sigma)\,\mathrm{d}\sigma, \quad \boldsymbol{\ell} \in \mathcal{K}.$$

Subsequently, the divergence-free profile $\overline{\boldsymbol{u}} \in \mathrm{W}^{1,2}((0,1);\mathrm{C}^\infty(\mathbb{T}^2;\mathbb{R}^2))$ is defined by

$$\overline{\boldsymbol{u}}(\cdot,t) := \overline{\boldsymbol{y}}(t) + \mathbb{I}_{[T_b,1]}(t)\overline{\boldsymbol{y}}^\star(\cdot, t-T_b), \tag{3.8}$$

with $\overline{\boldsymbol{y}}^{\star} \in \mathrm{W}^{1,2}((0,1); \mathrm{C}^{\infty}(\mathbb{T}^2; \mathbb{R}^2))$ being the "observable" vector field

$$\overline{\boldsymbol{y}}^{\star}(\boldsymbol{x},t) \coloneqq \sum_{\boldsymbol{\ell}\in\mathcal{K}} \left(\psi^{\mathrm{s}}_{\boldsymbol{\ell}}(t) s_{\boldsymbol{\ell}}(\boldsymbol{x})\boldsymbol{\ell}^{\perp} + \psi^{\mathrm{c}}_{\boldsymbol{\ell}}(t) c_{\boldsymbol{\ell}}(\boldsymbol{x})\boldsymbol{\ell}^{\perp}\right), \tag{3.9}$$

where $\boldsymbol{\ell}^{\perp} \coloneqq [-\ell_2, \ell_1]^{\top}$ for any $\boldsymbol{\ell} = [\ell_1, \ell_2]^{\top} \in \mathcal{K}$.

Remark 3.6. The vector field $\overline{\boldsymbol{u}}$, as described in (3.8) by means of $\overline{\boldsymbol{y}}$ and $\overline{\boldsymbol{y}}^{\star}$, serves multiple purposes. On the one hand, during the time interval (T_a, T_b), information travels in a specific way through the control region ω when following the integral curves of $\overline{\boldsymbol{y}}$. On the other hand, since $\overline{\boldsymbol{y}}^{\star}$ is defined using an observable family, one can obtain finite-dimensional, but non-localized, controls for associated linear transport equations.

Finally, keeping in mind our goal of squeezing the actions of finite-dimensional controls into the small control region ω, we introduce the flow map

$$\Xi(\boldsymbol{x},t) = [\Phi(\boldsymbol{x},t), \Psi(\boldsymbol{x},t)]^{\top} \coloneqq \boldsymbol{\mathcal{U}}\left(\boldsymbol{\mathcal{Y}}(\boldsymbol{x},t,0), 1, \overline{\tau}(t)\right), \tag{3.10}$$

where $\overline{\tau}$ is the function from (3.6) and $\boldsymbol{\mathcal{U}}$ denotes the solution to the system of ordinary differential equations

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\mathcal{U}}(\boldsymbol{x},s,t) = \overline{\boldsymbol{u}}(\boldsymbol{\mathcal{U}}(\boldsymbol{x},s,t),t), \\ \boldsymbol{\mathcal{U}}(\boldsymbol{x},s,s) = \boldsymbol{x}. \end{cases} \tag{3.11}$$

4 Saturation property

The collection $(s_{\boldsymbol{\ell}}, c_{\boldsymbol{\ell}})_{\boldsymbol{\ell}\in\mathbb{Z}^2_*}$ specified in (2.7) constitutes a complete orthogonal system in H^m for each $m \in \mathbb{N}$. For any given non-empty finite subset $\mathcal{E} \subset \mathbb{Z}^2_*$, we define a finite-dimensional space via

$$\mathcal{H}(\mathcal{E}) \coloneqq \operatorname{span}\left\{s_{\boldsymbol{\ell}}, c_{\boldsymbol{\ell}} \mid \boldsymbol{\ell} \in \mathcal{E}\right\}.$$

Remark 4.1. Since sin and cos are odd and even functions respectively, one has the relation $\mathcal{H}(\mathcal{E}) = \mathcal{H}(-\mathcal{E})$.

Based on the above choice of $\mathcal{E}$, we determine now a non-decreasing sequence of finite subsets $(\mathcal{E}_j)_{j\in\mathbb{N}_0} \subset \mathbb{Z}^2_*$ by taking $\mathcal{E}_0 \coloneqq \mathcal{E} \cup (-\mathcal{E})$ and

$$\mathcal{E}_j \coloneqq \mathcal{E}_{j-1} \cup \left\{\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2 \mid \boldsymbol{\ell}_1 \in \mathcal{E}_{j-1}, \boldsymbol{\ell}_2 \in \mathcal{E}_0, \boldsymbol{\ell}_1 \nparallel \boldsymbol{\ell}_2\right\}, \quad j \in \mathbb{N}, \tag{4.1}$$

where the constraint $\boldsymbol{\ell}_1 \nparallel \boldsymbol{\ell}_2$ ensures that each $\mathcal{E}_j$ is well-defined as a subset of $\mathbb{Z}^2_*$. The associated sequence of subspaces $(\mathcal{H}_j(\mathcal{E}))_{j\in\mathbb{N}_0} \subset \mathrm{H}^m$ is then introduced by

$$\mathcal{H}_j(\mathcal{E}) \coloneqq \mathcal{H}(\mathcal{E}_j), \ \ j \in \mathbb{N}_0.$$

We provide a simple characterization of subsets $\mathcal{E} \subset \mathbb{Z}^2_*$ for which $\cup_{i\in\mathbb{N}_0}\mathcal{H}_j(\mathcal{E})$ is dense in all H^m spaces.

Definition 4.2. We say that a finite subset $\mathcal{E} \subset \mathbb{Z}^2_*$ is

a) a generator if $\mathrm{span}_{\mathbb{Z}}(\mathcal{E}) = \mathbb{Z}^2$;

b) saturating if $\cup_{i\in\mathbb{N}_0}\mathcal{H}_i(\mathcal{E})$ is dense in H^m for each $m \in \mathbb{N}$.

The next two lemmas, whose proofs are obvious, are going to be used in the proofs of Section 5.

Lemma 4.3. *A finite subset $\mathcal{E} \subset \mathbb{Z}^2_*$ is saturating if and only if it is a generator.*

In notable contrast to previous literature such as [2,3], where saturating sets have been employed in the context of two-dimensional Navier–Stokes and Euler systems, no condition on the length of vectors belonging to the generator is required here. This simpler characterization of saturation is enabled by the different underlying strategy of this article, which is based on a linear test involving vector fields that are constructed from observable families.

Lemma 4.4. *For any finite subset $\mathcal{E} \subset \mathbb{Z}^2_*$, any integer $j \in \mathbb{N}$, and any vector $\boldsymbol{\ell} \in \mathcal{E}_j$, we have*

$$s_{\boldsymbol{\ell}} = s_{\boldsymbol{\ell}_1} c_{\boldsymbol{\ell}_2} + c_{\boldsymbol{\ell}_1} s_{\boldsymbol{\ell}_2}, \quad c_{\boldsymbol{\ell}} = c_{\boldsymbol{\ell}_1} c_{\boldsymbol{\ell}_2} - s_{\boldsymbol{\ell}_1} s_{\boldsymbol{\ell}_2},$$

where $\boldsymbol{\ell} = \boldsymbol{\ell}_1 + \boldsymbol{\ell}_2$ with $\boldsymbol{\ell}_1 \in \mathcal{E}_{j-1}$, $\boldsymbol{\ell}_2 \in \mathcal{E}_0$, and $\boldsymbol{\ell}_1 \nparallel \boldsymbol{\ell}_2$.

5 Controllability in small time via large localized controls

In this section, the proof of Theorem 2.3 is carried out. Our strategy comprises three steps. First, a controllability problem for a linearized inviscid system is solved with finite-dimensional controls supported in the whole torus $\mathbb{T}^2$. Subsequently, the approximate controllability for a modified linear problem is established using a control of the form (2.10). Ultimately, hydrodynamic scaling properties are exploited for passing to the nonlinear viscous system (2.1).

5.1 Finite-dimensional controls acting on the whole torus

We establish the approximate controllability for a transport equation driven by $\mathcal{H}_0(\mathcal{K})$-valued controls supported within the whole torus $\mathbb{T}^2$, where $\mathcal{K} \subset \mathbb{Z}^2_*$ is the generator that has been fixed in Section 2. As a reminder, the numbers $T_b \in (0, 1)$ and $T^\star = 1 - T_b$ have been defined in Section 3.3 and the vector field $\overline{\boldsymbol{u}}$ has been introduced in Section 3.4.

Theorem 5.1. *For any given $m \in \mathbb{N}$, $v_1 \in \mathrm{H}^m$, and $\varepsilon > 0$, there exists a control*

$$g \in \mathrm{L}^2((0,1);\mathcal{H}_0(\mathcal{K}))$$

such that the unique solution

$$v \in \mathrm{C}^0([0,1];\mathrm{H}^m) \cap \mathrm{W}^{1,2}((0,1);\mathrm{H}^{m-1})$$

to the linear transport problem

$$\partial_t v + (\overline{\boldsymbol{u}} \cdot \boldsymbol{\nabla})v = \mathbb{I}_{[T_b,1]}(t)g, \quad v(\cdot,0) = 0 \tag{5.1}$$

obeys at the terminal time $t = 1$ the estimate

$$\|v(\cdot,1) - v_1\|_m < \varepsilon. \tag{5.2}$$

Proof. The idea consists of reducing the controllability problem at hand to a version posed on the short time interval $[T_b,1]$, during which the definition of the vector field $\overline{\boldsymbol{u}}$ in (3.8) involves an observable family. Owing to the structure of $\overline{\boldsymbol{y}}^\star$, as described in (3.9), the existence of finite-dimensional controls is then accomplished by way of combining functional analytic tools with geometric arguments.

Step 1. Reduction. During the time interval $[0,T_b]$ we plug the zero control $g = 0$ into (5.1). Thus, since (5.1) is a homogeneous transport equation when considered on $[0,T_b]$, the terminal condition $v(\boldsymbol{x},T_b) = 0$ holds for all $\boldsymbol{x} \in \mathbb{T}^2$. During the remaining time interval $[T_b,1]$, we then seek a control $\widetilde{g} \in \mathrm{L}^2((T_b,1);\mathcal{H}_0(\mathcal{K}))$ solving in $\mathbb{T}^2 \times (T_b,1)$ the controllability problem

$$\partial_t v + (\overline{\boldsymbol{u}} \cdot \boldsymbol{\nabla})v = \widetilde{g}, \quad v(\cdot,T_b) = 0, \quad \|v(\cdot,1) - v_1\|_m < \varepsilon.$$

Bearing in mind the definition of $\overline{\boldsymbol{u}}$, the proof of Theorem 5.1 is reduced to the task of finding a control $\widehat{g} \in \mathrm{L}^2((0,T^\star);\mathcal{H}_0(\mathcal{K}))$ which guarantees that the unique solution $v \colon \mathbb{T}^2 \times [0,T^\star] \longrightarrow \mathbb{R}$ to the transport equation

$$\partial_t v + (\overline{\boldsymbol{y}}^\star \cdot \boldsymbol{\nabla})v = \widehat{g}, \quad v(\cdot,0) = 0 \tag{5.3}$$

obeys the terminal condition

$$\|v(\cdot,T^\star) - v_1\|_m < \varepsilon. \tag{5.4}$$

To this end, we employ a similar strategy as developed in [17, Section 3.3], thereby relying on the fact that $\overline{\boldsymbol{y}}^\star$ has been constructed based on an observable family. First, we denote by

$$R(t,\tau) \colon \mathrm{H}^m \longrightarrow \mathrm{H}^m, \quad \widetilde{v}_0 \longmapsto \widetilde{v}(t), \quad 0 \leq \tau \leq t \leq T^\star$$

the two-parameter family of resolving operators in H^m for the linear problem

$$\partial_t \widetilde{v} + (\overline{\boldsymbol{y}}^\star \cdot \boldsymbol{\nabla})\widetilde{v} = 0, \quad \widetilde{v}(\cdot,\tau) = \widetilde{v}_0.$$

Consequentially, the resolving operator $A = A_{T^\star}\colon \mathrm{L}^2((0,T^\star);\mathrm{H}^m) \longrightarrow \mathrm{H}^m$ for (5.3) at time $t = T^\star$ can be represented in the form

$$Ag = \int_0^{T^\star} R(T^\star,\tau)g(\tau)\,\mathrm{d}\tau, \quad g \in \mathrm{L}^2((0,T^\star);\mathrm{H}^m).$$

After denoting the orthogonal projector onto $\mathcal{H}_0(\mathcal{K})$ with respect to H^m as

$$P_{\mathcal{H}_0(\mathcal{K})}\colon \mathrm{H}^m \longrightarrow \mathcal{H}_0(\mathcal{K}) \subset \mathrm{H}^m,$$

we introduce the control-to-state operator

$$A_1 \coloneqq AP_{\mathcal{H}_0(\mathcal{K})}\colon \mathrm{L}^2((0,T^\star);\mathrm{H}^m) \longrightarrow \mathrm{H}^m.$$

In order to solve the controllability problem constituted by (5.3) and (5.4) with controls belonging to $\mathrm{L}^2((0,1);\mathcal{H}_0(\mathcal{K}))$, it suffices to demonstrate that the image of A_1 is dense in H^m. The latter property is equivalent to

$$\ker(A_1^*) = \{0\},$$

where the adjoint operator $A_1^*\colon \mathrm{H}^m \longrightarrow \mathrm{L}^2((0,T^\star);\mathrm{H}^m)$ can be expressed with the aid of the H^m-adjoint $R(T^\star,\tau)^*$ for $R(T^\star,\tau)$ as

$$(A_1^*z)(\tau) \coloneqq P_{\mathcal{H}_0(\mathcal{K})}R(T^\star,\tau)^*z, \quad \tau \in (0,T^\star).$$

Step 2. The idea for showing $\ker(A_1^*) = \{0\}$. Let $z \in \ker(A_1^*)$ be arbitrarily fixed. Then, for almost all $\tau \in [0,T^\star]$ and all $g \in \mathcal{H}_0(\mathcal{K})$, one has

$$\left\langle R(T^\star,\tau)g, z\right\rangle_m = \left\langle P_{\mathcal{H}_0(\mathcal{K})}g, R(T^\star,\tau)^*z\right\rangle_m = \left\langle g, (A_1^*z)(\tau)\right\rangle_m = 0.$$

In fact, because $\tau \longmapsto R(T^\star,\tau)$ is continuous, one even gets

$$\left\langle R(T^\star,\tau)g, z\right\rangle_m = 0 \tag{5.5}$$

for all $\tau \in [0,T^\star]$ and $g \in \mathcal{H}_0(\mathcal{K})$. Consequently, taking $\tau = T^\star$ in (5.5) yields

$$\langle g, z\rangle_m = 0 \tag{5.6}$$

for each $g \in \mathcal{H}_0(\mathcal{K})$. Therefore, z is orthogonal to $\mathcal{H}_0(\mathcal{K})$ in H^m. Since a dense subspace of a Hilbert space has a trivial orthogonal complement, and noting that by Lemma 4.3 the inclusion $\cup_{i\in\mathbb{N}_0}\mathcal{H}_i(\mathcal{K}) \subset \mathrm{H}^m$ is dense, it only remains to establish orthogonality relations of the type (5.6) for all $g \in \cup_{i\in\mathbb{N}_0}\mathcal{H}_i(\mathcal{K})$ in order to conclude that $z = 0$.

Given any $T_1 \in [0, T^\star]$, let us distinguish the element $z_1 \coloneqq R(T^\star, T_1)^* z$, which satisfies due to (5.5) the relations

$$\langle R(T_1,\tau)g, z_1\rangle_m = \left\langle R(T^\star,\tau)g, z\right\rangle_m = 0 \tag{5.7}$$

for all $\tau \in [0, T_1]$ and $g \in \mathcal{H}_0(\mathcal{K})$. Taking $\tau = T_1$ in (5.7), we find that

$$\langle g, z_1\rangle_m = \langle R(T_1,T_1)g, z_1\rangle_m = 0 \tag{5.8}$$

for all $g \in \mathcal{H}_0(\mathcal{K})$. If the equality $\langle g, z_1\rangle_m = 0$ from (5.8) would be valid for all $g \in \mathcal{H}_i(\mathcal{K})$ with some $i \in \mathbb{N}$, then by inserting $T_1 = T^\star$ into the definition of z_1, also (5.6) would hold for all $g \in \mathcal{H}_i(\mathcal{K})$. Accordingly, by induction over i, we subsequently show that

$$\langle g, z_1\rangle_m = \langle R(T_1,T_1)g, z_1\rangle_m = 0, \quad g \in \mathcal{H}_i(\mathcal{K}) \tag{5.9}$$

for arbitrary $i \in \mathbb{N}$.

Step 3. Induction base. We begin with establishing (5.9) for $i = 1$. To this end, we arbitrarily fix $g \in \mathcal{H}_0(\mathcal{K})$ and consider

$$q(t,\tau) \coloneqq R(t+\tau,\tau)g, \quad Q(t,\tau) \coloneqq \partial_\tau q(t,\tau), \quad 0 \leq t \leq T^\star - \tau.$$

Viewing q and Q as functions of $t \in [0, T^\star - \tau]$, which take values in H^m and depend on a parameter $\tau \in [0, T^\star)$, we observe that they solve for all $(\boldsymbol{x}, t) \in \mathbb{T}^2 \times (0, T^\star - \tau)$ the initial value problems

$$\begin{cases}\partial_t q(\boldsymbol{x},t,\tau) + (\overline{\boldsymbol{y}}^\star(\boldsymbol{x},t+\tau)\cdot\boldsymbol{\nabla})q(\boldsymbol{x},t,\tau) = 0,\\ \partial_t Q(\boldsymbol{x},t,\tau) + (\overline{\boldsymbol{y}}^\star(\boldsymbol{x},t+\tau)\cdot\boldsymbol{\nabla})Q(\boldsymbol{x},t,\tau) + (\partial_t\overline{\boldsymbol{y}}^\star(\boldsymbol{x},t+\tau)\cdot\boldsymbol{\nabla})q(\boldsymbol{x},t,\tau) = 0,\\ q(\boldsymbol{x},0,\tau) = g,\\ Q(\cdot,0,\tau) = 0.\end{cases}$$

By integrating the H^m-inner product of the equation for Q with z_1 from $t = 0$ to $t = T_1 - \tau$, it follows that

$$\begin{aligned}0 = \langle Q(\boldsymbol{x}, T_1-\tau,\tau), z_1\rangle_m &+ \int_0^{T_1-\tau} \left\langle(\overline{\boldsymbol{y}}^\star(\cdot,t+\tau)\cdot\boldsymbol{\nabla})Q(\cdot,t,\tau), z_1\right\rangle_m \,\mathrm{d}t\\ &+ \int_0^{T_1-\tau}\left\langle(\partial_t\overline{\boldsymbol{y}}^\star(\cdot,t+\tau)\cdot\boldsymbol{\nabla})q(\boldsymbol{x},t,\tau), z_1\right\rangle_m \,\mathrm{d}t,\end{aligned}$$

which implies

$$0 = \langle Q(\boldsymbol{x}, T_1 - \tau, \tau), z_1\rangle_m + \int_0^{T_1-\tau} \left\langle (\overline{\boldsymbol{y}}^\star(\cdot, t+\tau)\cdot\boldsymbol{\nabla})Q(\cdot, t, \tau), z_1\right\rangle_m \,\mathrm{d}t \\ + \int_\tau^{T_1} \left\langle (\partial_t\overline{\boldsymbol{y}}^\star(\cdot, t)\cdot\boldsymbol{\nabla})[R(t,\tau)g], z_1\right\rangle_m \,\mathrm{d}t. \tag{5.10}$$

Moreover, by calculating $\partial_\tau(R(t+\tau,\tau)g)$ using the chain rule, and evaluating the result at $t = T_1 - \tau$, we find that

$$\partial_\tau R(t,\tau)g_{|_{t=T_1}}(\boldsymbol{x}) = Q(\boldsymbol{x}, T_1-\tau, \tau) + (\overline{\boldsymbol{y}}^\star(\boldsymbol{x}, T_1)\cdot\boldsymbol{\nabla})[R(T_1,\tau)g](\boldsymbol{x}). \tag{5.11}$$

As a result, after taking ∂_τ in $\langle R(T_1,\tau)g, z_1\rangle_m = 0$ from (5.7) and plugging (5.11) into (5.10), we get

$$0 = \int_0^{T_1-\tau} \left\langle (\overline{\boldsymbol{y}}^\star(\cdot, t+\tau)\cdot\boldsymbol{\nabla})Q(\cdot, t, \tau), z_1\right\rangle_m \,\mathrm{d}t \\ + \int_\tau^{T_1} \left\langle (\partial_t\overline{\boldsymbol{y}}^\star(\cdot, t)\cdot\boldsymbol{\nabla})[R(t,\tau)g], z_1\right\rangle_m \,\mathrm{d}t - \left\langle (\overline{\boldsymbol{y}}^\star(\cdot, T_1)\cdot\boldsymbol{\nabla})[R(T_1,\tau)g], z_1\right\rangle_m .$$

Differentiating the latter relation with respect to τ, while invoking the definition of the vector field $\overline{\boldsymbol{y}}^\star$ from (3.9), yields

$$b(\tau) + \sum_{\ell\in\mathcal{K}} \left(a^s_{\boldsymbol{\ell}}(\tau)\phi^s_{\boldsymbol{\ell}}(\tau) + a^c_{\boldsymbol{\ell}}(\tau)\phi^c_{\boldsymbol{\ell}}(\tau)\right) = 0, \tag{5.12}$$

where $b \in \mathrm{C}^0([0,T_1];\mathbb{R})$ is the function

$$b(\tau) := \partial_\tau \int_0^{T_1-\tau} \left\langle (\overline{\boldsymbol{y}}^\star(\cdot, t+\tau)\cdot\boldsymbol{\nabla})Q(\cdot, t, \tau), z_1\right\rangle_m \,\mathrm{d}t \\ - \sum_{\boldsymbol{\ell}\in\mathcal{K}} \int_0^\tau \left\langle \left(\partial_t\phi(\tau)\int_0^t \left(\phi^s_{\boldsymbol{\ell}}(\sigma)s_{\boldsymbol{\ell}}\boldsymbol{\ell}^\perp + \phi^c_{\boldsymbol{\ell}}(\sigma)c_{\boldsymbol{\ell}}\boldsymbol{\ell}^\perp\right)\mathrm{d}\sigma\cdot\boldsymbol{\nabla}\right) g, z_1\right\rangle_m \,\mathrm{d}t \\ + \int_\tau^{T_1} \left\langle (\partial_t\overline{\boldsymbol{y}}^\star(\cdot, t)\cdot\boldsymbol{\nabla})[\partial_\tau R(t,\tau)g], z_1\right\rangle_m \,\mathrm{d}t \\ - \partial_\tau \left\langle (\overline{\boldsymbol{y}}^\star(\cdot, T_1)\cdot\boldsymbol{\nabla})[R(T_1,\tau)g], z_1\right\rangle_m ,$$

and the coefficients $(a^s_{\boldsymbol{\ell}}, a^c_{\boldsymbol{\ell}})_{\boldsymbol{\ell}\in\mathcal{K}} \subset \mathrm{C}^1([0,T^\star];\mathbb{R})$ are given by

$$a^s_{\boldsymbol{\ell}}(\tau) := -\phi(\tau)\left\langle (s_{\boldsymbol{\ell}}\boldsymbol{\ell}^\perp\cdot\boldsymbol{\nabla})g, z_1\right\rangle_m, \quad a^c_{\boldsymbol{\ell}}(\tau) := -\phi(\tau)\left\langle (c_{\boldsymbol{\ell}}\boldsymbol{\ell}^\perp\cdot\boldsymbol{\nabla})g, z_1\right\rangle_m .$$

Since the family $(\phi^s_{\boldsymbol{\ell}}, \phi^c_{\boldsymbol{\ell}})_{\boldsymbol{\ell}\in\mathcal{K}}$ is observable, the relation (5.12) yields

$$\left\langle (s_{\boldsymbol{\ell}}\boldsymbol{\ell}^\perp\cdot\boldsymbol{\nabla})g, z_1\right\rangle_m = \left\langle (c_{\boldsymbol{\ell}}\boldsymbol{\ell}^\perp\cdot\boldsymbol{\nabla})g, z_1\right\rangle_m = 0, \quad \boldsymbol{\ell}\in\mathcal{K}. \tag{5.13}$$

Now, we take an arbitrary element $\xi \in \mathcal{H}_1(\mathcal{K})$ and without loss of generality assume that for $\boldsymbol{\ell}_1, \boldsymbol{\ell}_2 \in \mathcal{K}$ with $\boldsymbol{\ell}_1 \not\parallel \boldsymbol{\ell}_2$ one has either $\xi = s_{\boldsymbol{\ell}_1+\boldsymbol{\ell}_2}$ or $\xi = c_{\boldsymbol{\ell}_1+\boldsymbol{\ell}_2}$. In view of Lemma 4.4, the function ξ admits one of the two representations

$$\xi = s_{\boldsymbol{\ell}_1} c_{\boldsymbol{\ell}_2} + c_{\boldsymbol{\ell}_1} s_{\boldsymbol{\ell}_2}, \quad \xi = c_{\boldsymbol{\ell}_1} c_{\boldsymbol{\ell}_2} - s_{\boldsymbol{\ell}_1} s_{\boldsymbol{\ell}_2}.$$

Then, by choosing $g = c_{\boldsymbol{\ell}_1}$ or $g = s_{\boldsymbol{\ell}_1}$ in (5.13), we obtain the relation $\langle \xi, z\rangle_m = 0$, which implies (5.9) for $i = 1$ due to the arbitrariness of $\xi \in \mathcal{H}_1(\mathcal{K})$.

Step 4. Induction step. With the intention of closing the induction argument, we assume now for arbitrarily fixed $i \in \mathbb{N}$ that

$$\langle g, z\rangle_m = 0, \quad g \in \mathcal{H}_i(\mathcal{K}). \tag{5.14}$$

By analysis similar to the previous step, the statement in (5.14) leads to

$$\left\langle (s_{\boldsymbol{\ell}} \boldsymbol{\ell}^{\perp} \cdot \boldsymbol{\nabla}) g, z_1 \right\rangle_m = \left\langle (c_{\boldsymbol{\ell}} \boldsymbol{\ell}^{\perp} \cdot \boldsymbol{\nabla}) g, z_1 \right\rangle_m = 0, \quad \boldsymbol{\ell} \in \mathcal{K}, \quad g \in \mathcal{H}_i(\mathcal{K}). \tag{5.15}$$

It remains to show that z_1 is orthogonal to $\mathcal{H}_{i+1}(\mathcal{K})$. To this end, let the sequence $(\mathcal{K}_j)_{j\in\mathbb{N}_0}$ be defined as in (4.1). Using the relations (5.15), wherein $\boldsymbol{\ell} = \boldsymbol{\ell}_2 \in \mathcal{K}$ and $g = s_{\boldsymbol{\ell}_1}$ or $g = c_{\boldsymbol{\ell}_1}$ with $\boldsymbol{\ell}_1 \in \mathcal{K}_i$ such that $\boldsymbol{\ell}_1 \not\parallel \boldsymbol{\ell}_2$, and applying Lemma 4.4, we observe that z is orthogonal to $s_{\boldsymbol{\ell}}$ and $c_{\boldsymbol{\ell}}$ with any $\boldsymbol{\ell} \in \mathcal{K}_{i+1}$. This implies that $\langle g, z\rangle_m = 0$ for all $g \in \mathcal{H}_{i+1}(\mathcal{K})$ and completes the proof of the theorem. □

Remark 5.2. One can infer from Theorem 5.1 that, for fixed $\varepsilon > 0$, there exists a bounded linear operator $C_\varepsilon \colon \mathrm{H}^m \longrightarrow \mathrm{L}^2((0,1);\mathcal{H}_0)$ assigning to any given target state v_1 from a bounded subset of H^m a control $g \in \mathrm{L}^2((0,1);\mathcal{H}_0)$ such that the corresponding solution v to (5.1) satisfies $\|v(\cdot,1) - v_1\|_{m-1} < \varepsilon$. To justify this, we introduce the resolving operator

$$\mathcal{A} \colon \mathrm{H}^m \times \mathrm{L}^2((0,1),\mathcal{H}_0(\mathcal{K})) \longrightarrow \mathrm{C}^0([0,1];\mathrm{H}^m) \cap \mathrm{W}^{1,2}((0,1);\mathrm{H}^{m-1})$$

which associates with $v_0 \in \mathrm{H}^m$ and $g \in \mathrm{L}^2((0,1);\mathcal{H}_0)$ the solution v to the problem

$$\partial_t v + (\overline{\boldsymbol{u}} \cdot \boldsymbol{\nabla}) v = \mathbb{I}_{[T_b,1]}(t) g, \quad v(\cdot,0) = v_0,$$

while denoting by $\mathcal{A}_1$ the restriction of $\mathcal{A}$ to the terminal time $t = 1$. As a result of Theorem 5.1, the range of the linear operator

$$\mathcal{A}_1(0,\cdot) \colon \mathrm{L}^2((0,1),\mathcal{H}_0(\mathcal{K})) \longrightarrow \mathrm{H}^m$$

is dense in H^m. Therefore, by [13, Proposition 2.6], while also referring to the proof of Theorem 2.3 in [17], there exists a bounded linear approximate right inverse C_ε for $\mathcal{A}_1(0,\cdot)$, which is as desired.

5.2 Localizing the controls

As a consequence of Theorem 5.1, in what follows, we achieve the global approximate controllability of a linear transport equation driven by localized degenerate controls. At this point, in order to prepare for the transition to the nonlinear problem discussed in Section 5.3, the convection in the considered transport equation has to take place along the vector field $\overline{\boldsymbol{y}}$ constructed in Section 3.3. The underlying idea is related to the return method (*cf.* [9, Part 2, Chapter 6]) in the sense that the profile $\overline{\boldsymbol{y}}$ is nontrivial, vanishes near $t \in \{0, 1\}$, and the associated flow transports each particle through the control region. Below, $\widetilde{\eta}_{(\zeta_{\boldsymbol{\ell}}^s, \zeta_{\boldsymbol{\ell}}^c)_{\boldsymbol{\ell}\in\mathcal{K}}}$ refers to the average-free force described in (2.10) for a choice of control parameters $(\zeta_{\boldsymbol{\ell}}^s, \zeta_{\boldsymbol{\ell}}^c)_{\boldsymbol{\ell}\in\mathcal{K}}$.

Theorem 5.3. *For any $m \in \mathbb{N}$, $\varepsilon > 0$, and $v_1 \in \mathrm{H}^m$, there exist control parameters*

$$(\zeta_{\boldsymbol{\ell}}^s, \zeta_{\boldsymbol{\ell}}^c)_{\boldsymbol{\ell}\in\mathcal{K}} \subset \mathrm{L}^2((0,1);\mathbb{R})$$

such that the unique function $v \in \mathrm{C}^0([0,1];\mathrm{H}^m) \cap \mathrm{W}^{1,2}((0,1);\mathrm{H}^{m-1})$ which solves the transport equation

$$\partial_t v + (\overline{\boldsymbol{y}} \cdot \boldsymbol{\nabla}) v = \widetilde{\eta}_{(\zeta_{\boldsymbol{\ell}}^s, \zeta_{\boldsymbol{\ell}}^c)_{\boldsymbol{\ell}\in\mathcal{K}}}, \quad v(\cdot, 0) = 0 \tag{5.16}$$

obeys at the terminal time $t = 1$ the estimate

$$\|v(\cdot, 1) - v_1\|_m < \varepsilon. \tag{5.17}$$

Proof. With the help of Theorem 5.1, the control parameters $(\zeta_{\boldsymbol{\ell}}^s, \zeta_{\boldsymbol{\ell}}^c)_{\boldsymbol{\ell}\in\mathcal{K}}$ are determined from a finite-dimensional control supported in the whole torus $\mathbb{T}^2$. Subsequently, by means of the method of characteristics and the particular constructions carried out in Section 3, the action of the global force is squeezed into the possibly small control region ω. Meanwhile, as anticipated by (5.16), convection occurs now along the profile $\overline{\boldsymbol{y}}$ provided by Theorem 3.3.

Step 1. Determining the control parameters. By Theorem 5.1, there exists a finite-dimensional control $g \in \mathrm{L}^2((0,1);\mathcal{H}_0(\mathcal{K}))$ such that the solution

$$\widetilde{v} \in \mathrm{C}^0([0,1];\mathrm{H}^m) \cap \mathrm{W}^{1,2}((0,1);\mathrm{H}^{m-1})$$

to the transport equation (5.1) satisfies

$$\|\widetilde{v}(\cdot, 1) - v_1\|_m < \varepsilon.$$

As a consequence, there are uniquely determined coefficients

$$(\zeta_{\boldsymbol{\ell}}^s, \zeta_{\boldsymbol{\ell}}^c)_{\boldsymbol{\ell}\in\mathcal{K}} \subset \mathrm{L}^2((0,1);\mathbb{R})$$

such that g is represented by means of the linear combination

$$g(\boldsymbol{x},t) = \sum_{\boldsymbol{\ell}\in\mathcal{K}} \left[\zeta_{\boldsymbol{\ell}}^{s}(t)s_{\boldsymbol{\ell}}(\boldsymbol{x}) + \zeta_{\boldsymbol{\ell}}^{c}(t)c_{\boldsymbol{\ell}}(\boldsymbol{x})\right]. \tag{5.18}$$

Step 2. Analysis of characteristic curves. Let v be the solution to (5.16) with respect to the control parameters $(\zeta_{\boldsymbol{\ell}}^{s}, \zeta_{\boldsymbol{\ell}}^{c})_{\boldsymbol{\ell}\in\mathcal{K}}$ found during the previous step. Furthermore, $\boldsymbol{\mathcal{Y}}$ denotes the flow associated via (3.7) with the profile $\overline{\boldsymbol{y}}$ fixed in Section 3.4. Then, the property P2 from Section 3.3, together with the method of characteristics, gives for any $\boldsymbol{x} \in \mathbb{T}^2$ rise to the representation

$$v(\boldsymbol{x},1) = v(\boldsymbol{\mathcal{Y}}(\boldsymbol{x},0,1),1) = \int_0^1 \widetilde{\eta}_{(\zeta_{\boldsymbol{\ell}}^{s},\zeta_{\boldsymbol{\ell}}^{c})_{\boldsymbol{\ell}\in\mathcal{K}}}(\boldsymbol{\mathcal{Y}}(\boldsymbol{x},0,s),s)\,\mathrm{d}s.$$

We shall verify the estimate (5.17) by utilizing the known characteristic curves associated with the transport equations (5.1) and (5.16) respectively. Hereto, we arbitrarily fix $\boldsymbol{x} \in \mathbb{T}^2$ and define $\boldsymbol{q} := \boldsymbol{\mathcal{U}}(\boldsymbol{x},0,1)$, where the flow $\boldsymbol{\mathcal{U}}$ is defined in (3.11). Inserting the partition of unity $(\mu_l)_{l\in\{1,\dots,K\}}$ from (3.4) into the solution formula for (5.1) provides

$$\begin{aligned}\widetilde{v}(\boldsymbol{q},1) &= \int_0^1 \mathbb{I}_{[T_b,1]}(s)g(\boldsymbol{\mathcal{U}}(\boldsymbol{x},0,s),s)\,\mathrm{d}s = \int_{T_b}^1 g(\boldsymbol{\mathcal{U}}(\boldsymbol{x},0,s),s)\,\mathrm{d}s\\ &= \sum_{l=1}^{K}\int_{T_b}^1 \mu_l(\boldsymbol{q})g(\boldsymbol{\mathcal{U}}(\boldsymbol{x},0,s),s)\,\mathrm{d}s.\end{aligned} \tag{5.19}$$

In view of (3.6), and for each $l \in \{1,\dots,K\}$, a change of variables yields

$$\begin{aligned}&\int_{T_b}^1 \mu_l(\boldsymbol{q})g(\boldsymbol{\mathcal{U}}(\boldsymbol{x},0,s),s)\,\mathrm{d}s\\ &\quad= \int_{t_a^l}^{t_b^l} \mu_l(\boldsymbol{q})g\left(\boldsymbol{\mathcal{U}}\left(\boldsymbol{x},0,\left(T_b+s-t_a^l\right)\right),\left(T_b+s-t_a^l\right)\right)\,\mathrm{d}s,\\ &\quad= \int_{t_a^l}^{t_b^l} \mu_l(\boldsymbol{q})g\left(\boldsymbol{\mathcal{U}}\left(\boldsymbol{x},0,\tau_l(s)\right),\tau_l(s)\right)\,\mathrm{d}s.\end{aligned} \tag{5.20}$$

By further utilizing in (5.20) the property P3 satisfied by $\overline{\boldsymbol{y}}$ and the corresponding flow $\boldsymbol{\mathcal{Y}}$, while recalling the definition of χ from (3.3), one has for each $l \in \{1,\dots,K\}$ that (*cf.* Figure 6)

$$\begin{aligned}&\int_{T_b}^1 \mu_l(\boldsymbol{q})g(\boldsymbol{\mathcal{U}}(\boldsymbol{x},0,s),s)\,\mathrm{d}s\\ &\qquad\qquad= \int_{t_a^l}^{t_b^l} \chi(\boldsymbol{\mathcal{Y}}(\boldsymbol{q},0,s))g\left(\boldsymbol{\mathcal{U}}\left(\boldsymbol{x},0,\tau_l(s)\right),\tau_l(s)\right)\,\mathrm{d}s.\end{aligned} \tag{5.21}$$

Thus, combining the identities (5.19) and (5.21) leads to

$$\widetilde{v}(\boldsymbol{q},1) = \sum_{l=1}^{K} \int_{t_a^l}^{t_b^l} \chi(\boldsymbol{\mathcal{Y}}(\boldsymbol{q},0,s)) g\left(\boldsymbol{\mathcal{U}}\left(\boldsymbol{x},0,\tau_l(s)\right),\tau_l(s)\right) \mathrm{d}s. \tag{5.22}$$

Since $\boldsymbol{\mathcal{U}}(\cdot,s,t)$ is for each $s,t \in [0,1]$ a homeomorphism of $\mathbb{T}^2$, and by employing (2.9), (3.10), (5.18), and (5.22), we infer that

$$\widetilde{v}(\boldsymbol{x},1) = \int_0^1 \widehat{\eta}_{(\zeta_\ell^s,\zeta_\ell^c)_{\ell\in\mathcal{K}}}(\boldsymbol{\mathcal{Y}}(\boldsymbol{x},0,s),s)\,\mathrm{d}s \tag{5.23}$$

for all $\boldsymbol{x} \in \mathbb{T}^2$, which ensures that (5.17) holds when $\widetilde{\eta}$ is replaced by $\widehat{\eta}$ in (5.16).

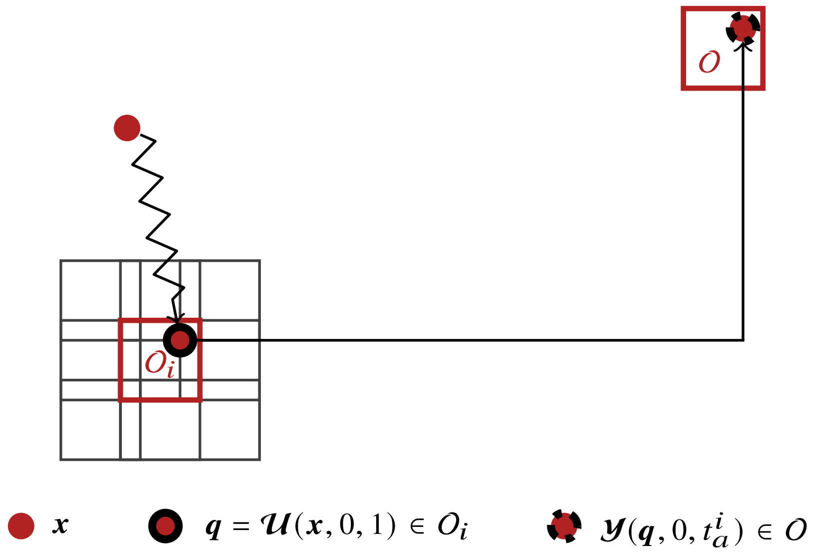


Figure 6: A sketch of several ideas related to the identities (5.20) and (5.21), referring to Section 3.1 for the notations. In the particular example displayed here, the point $\boldsymbol{q} \in O_i$ is not located in the interior of $O_i \cap O_l$ for any $l \in \{1,\dots,K\}\setminus\{i\}$, thus the integral over $[0,1]$ of the force $g(\boldsymbol{\mathcal{U}}(\boldsymbol{x},0,s))$ can be compressed into an integral over (t_a^i,t_b^i). During this short interval, the square O_i has already been moved by $\boldsymbol{\mathcal{Y}}$ into O and the flow $\boldsymbol{\mathcal{Y}}$ pauses. The overlapping squares at the bottom left indicate all members of the family $(O_l)_{l\in\{1,\dots,K\}}$ which intersect non-trivially with O_i.

Step 3. Using an average-free control. It remains to achieve the identity (5.23) with $\widetilde{\eta} = \widetilde{\eta}_{(\zeta_\ell^s,\zeta_\ell^c)_{\ell\in\mathcal{K}}}$ as defined in (2.10) instead of $\widehat{\eta} = \widehat{\eta}_{(\zeta_\ell^s,\zeta_\ell^c)_{\ell\in\mathcal{K}}}$. At the outset, since $g \in \mathrm{L}^2((0,1);\mathcal{H}_0(\mathcal{K}))$, we recognize that

$$\int_{\mathbb{T}^2} g(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x} = 0, \qquad \text{for a.a. } t \in [0,1]. \tag{5.24}$$

Because the vector field $\overline{\boldsymbol{u}}$ is divergence-free, we can further infer from (5.24) that the zero average of the initial data is preserved during the evolution governed by the transport equation (5.1), hence

$$\forall t \in [0,1]\colon \int_{\mathbb{T}^2} \widetilde{v}(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x} = 0.$$

Consequentially, because (5.23) is true for all $\boldsymbol{x} \in \mathbb{T}^2$, one observes that

$$0 = \int_{\mathbb{T}^2} \widetilde{v}(z,1)\,\mathrm{d}z = \int_{\mathbb{T}^2}\int_0^1 \widehat{\eta}(\boldsymbol{\mathcal{Y}}(z,0,s),s)\,\mathrm{d}s\,\mathrm{d}z. \tag{5.25}$$

In turn, as the homeomorphism $\boldsymbol{\mathcal{Y}}$ is measure-preserving, the relation (5.25) yields

$$0 = \int_0^1\int_{\mathbb{T}^2} \widehat{\eta}(z,s)\,\mathrm{d}z\,\mathrm{d}s. \tag{5.26}$$

Now, we arbitrarily fix $\boldsymbol{x} \in \mathbb{T}^2$ and select any $i \in \{1,\dots,K\}$ with $\boldsymbol{x} \in O_i$, where the family $(O_l)_{l\in\{1,\dots,K\}}$ is given in Section 3.1. Due to $\operatorname{supp}(\widetilde{\chi}) \subset \{\chi = 1\}$, as stated in Section 3.2, one has either $\boldsymbol{x} \notin \bigcup_{l\neq i} O_l$ or $\widetilde{\chi}(\boldsymbol{\mathcal{Y}}(\boldsymbol{x},0,t_a^i)) = 0$. Moreover, the property P3 from Section 3.3 renders $\boldsymbol{\mathcal{Y}}$ stationary on $[t_a^l, t_b^l]$ for each $l \in \{1,\dots,K\}$. Therefore, in the case $i \neq 1$, direct cancellations imply (*cf.* Remark 5.4)

$$\begin{aligned}
-\sum_{l=1}^{K}\sum_{j=1}^{l}\int_0^1 \mathbb{I}_{[t_a^l,t_b^l]}(s)\widetilde{\chi}_l(\boldsymbol{\mathcal{Y}}(\boldsymbol{x},0,s))\int_{\mathbb{T}^2}\widehat{\eta}(z,s-3(j-1)T^{\star})\,\mathrm{d}z\,\mathrm{d}s&\\
+\sum_{l=2}^{K}\sum_{k=1}^{l-1}\int_0^1 \mathbb{I}_{[t_a^l,t_b^l]}(s)\widetilde{\chi}(\boldsymbol{\mathcal{Y}}(\boldsymbol{x},0,s))\int_{\mathbb{T}^2}\widehat{\eta}(z,s-3kT^{\star})\,\mathrm{d}z\mathrm{d}s &= 0,
\end{aligned} \tag{5.27}$$

where $(\widetilde{\chi}_l)_{l\in\{1,\dots,K\}}$ are loosely speaking the cutoff functions (*cf.* (3.5))

$$\widetilde{\chi}_l = \begin{cases} \widetilde{\chi}_{\nearrow} = \text{top-right shift of } \widetilde{\chi}, & \text{if } l \text{ is a multiple of } \sqrt{K},\\ \widetilde{\chi}_{\rightarrow} = \text{right shift of } \widetilde{\chi}, & \text{otherwise.}\end{cases}$$

When $i = 1$, as illustrated in Figure 7, we can employ (5.26) in order to infer

$$\sum_{j=1}^{K}\int_0^1 \mathbb{I}_{[t_a^K,t_b^K]}(s)\widetilde{\chi}_K(\boldsymbol{\mathcal{Y}}(\boldsymbol{x},0,s))\int_{\mathbb{T}^2}\widehat{\eta}(z,s-3(j-1)T^{\star})\,\mathrm{d}z\,\mathrm{d}s = 0, \tag{5.28}$$

which yields (5.27), as well. In light of (2.10), (5.23), and (5.27), we obtain

$$\widetilde{v}(\boldsymbol{x},1) = \int_0^1 \widetilde{\eta}_{(\zeta_\ell^s,\zeta_\ell^c)_{\ell\in\mathcal{K}}}(\boldsymbol{\mathcal{Y}}(\boldsymbol{x},0,s),s)\,\mathrm{d}s,$$

which completes the proof. □

Remark 5.4. For the purpose of describing the intuition behind (5.27) and (5.28), we fix any $l \in \{1, \ldots, K\}$. The inner region of O_l that has an empty intersection with O_i for all $i \in \{1, \ldots, K\} \setminus \{l\}$ is called the "non-overlapping part" of O_l. Now, let us look at the situation where the square O_l has been moved by the flow $\boldsymbol{\mathcal{Y}}$ into the reference square O at the time $t = t_a^l$ and pauses there until $t = t_b^l$. During this short time interval, the contributions of $\fint_{\mathbb{T}^2} \widehat{\eta}$ only influence information transported by $\boldsymbol{\mathcal{Y}}$ from O_l. In order to define an average-free version $\widetilde{\eta}$ of the control $\widehat{\eta}$, we cannot simply subtract the latter contributions in O during $[t_a^l, t_b^l]$, as the result might fail to be a suitable control. Instead, $\fint_{\mathbb{T}^2} \widehat{\eta}$ is offset in $\operatorname{supp}(\widetilde{\chi}_l)$, which by definition corresponds either to the support of $\widetilde{\chi}_{\nearrow}$ or the support of $\widetilde{\chi}_{\rightarrow}$, depending on whether the square O_{l+1} is located top-right or right of O_l. However, by means of this strategy, if $l > 1$, previous average corrections from the time interval $[t_a^{l-1}, t_b^{l-1}]$ are already transported along with the non-overlapping part of O_l under $\boldsymbol{\mathcal{Y}}$. These are now, that is during $[t_a^l, t_b^l]$, erased from $\operatorname{supp}(\widetilde{\chi})$ and instead written to $\operatorname{supp}(\widetilde{\chi}_l)$. In this way, the functionality of the control $\widehat{\eta}$ is retained. When $l = K$, all the previous average corrections, and those made during $[t_a^K, t_b^K]$, accumulate at $t = t_b^K$ on parcels originating from the non-overlapping part of O_K under the flow $\boldsymbol{\mathcal{Y}}$, eventually canceling each other out in the time average taken over $[0, 1]$.

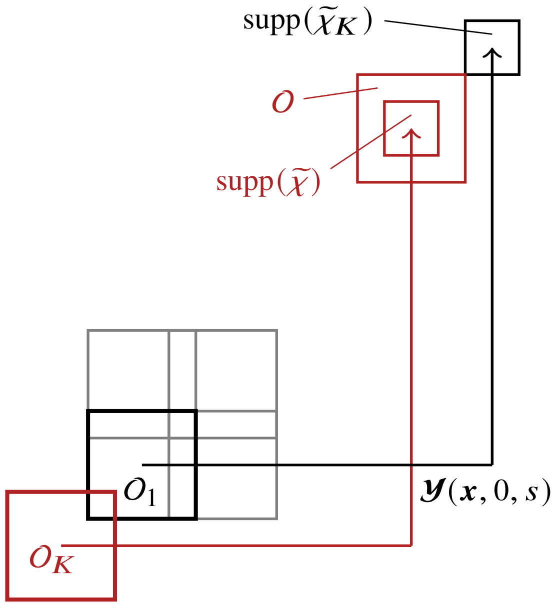


Figure 7: A schematic illustration of the case $\boldsymbol{x} \in O_1$ in (5.27). Hereto, note that $\widetilde{\chi}_K = \widetilde{\chi}_{\nearrow}$ (*cf.* (3.5)). Then, during the time interval $[t_a^K, t_b^K]$, the information originating from the non-overlapping part of O_1 has been transported by the flow $\boldsymbol{\mathcal{Y}}$ to the respective region containing $\operatorname{supp}(\widetilde{\chi}_K)$. The latter is located top-right of the reference square O. Meanwhile, the information from the non-overlapping part of O_K has been moved by $\boldsymbol{\mathcal{Y}}$ to the center region of the reference square O containing $\operatorname{supp}(\widetilde{\chi})$. The average corrections made during $[0, t_b^K]$ accumulate in $\operatorname{supp}(\widetilde{\chi})$ during $[t_a^K, t_b^K]$.

5.3 Passing to the nonlinear system

The approximate controllability result provided by Theorem 5.3 can now be translated back to the nonlinear viscous Navier–Stokes system in vorticity form (2.1). To this end, inspired by [7] and [17, Proposition 2.2], we first show how to control the vorticity equation during a short time interval by means of large controls.

Lemma 5.5. *For $m \in \mathbb{N}$, let any state $w_0 \in \mathrm{H}^{m+1}$, any force $h \in \mathrm{L}^2((0,1);\mathrm{H}^{m-1})$, and arbitrary parameters $(\rho_l)_{l\in\{1,\dots,N\}} \subset \mathrm{L}^2((0,1);\mathbb{R})$ be fixed. Moreover,*

- *denote by $v \in \mathrm{C}^0([0,1];\mathrm{H}^{m+1}) \cap \mathrm{W}^{1,2}((0,1);\mathrm{H}^m)$ the solution to the linear transport problem*

$$\partial_t v + (\overline{\boldsymbol{y}} \cdot \boldsymbol{\nabla})v = \widetilde{\eta}_{\rho_1,\dots,\rho_N}, \quad v(\cdot,0) = w_0, \tag{5.29}$$

  where the force $\widetilde{\eta} = \widetilde{\eta}_{\rho_1,\dots,\rho_N}$ is given via (2.10) in terms of $(\rho_l)_{l\in\{1,\dots,N\}}$;

- *for any given $\delta \in (0,1)$ and $t \in (0,\delta)$ consider the scaled control force*

$$\widetilde{\eta}_\delta(\cdot,t) = \widetilde{\eta}_{\delta,\rho_1,\dots,\rho_N}(\cdot,t) := \delta^{-1}\widetilde{\eta}_{\rho_1,\dots,\rho_N}(\cdot,\delta^{-1}t),$$

  *together with the scaled average (*cf. (2.14)*)*

$$\widetilde{\aleph}_\delta(t) = \int_{\mathbb{T}^2} \left(\overline{\boldsymbol{y}}_\delta(t) + \boldsymbol{U}_\delta(t)\right) \,\mathrm{d}\boldsymbol{x}, \quad \overline{\boldsymbol{y}}_\delta(t) := \delta^{-1}\overline{\boldsymbol{y}}(\delta^{-1}t), \quad \boldsymbol{U}_\delta(t) := \boldsymbol{U}(\delta^{-1}t).$$

Then, one has the convergence

$$\|S_\delta(w_0, h + \widetilde{\eta}_\delta, \widetilde{\aleph}_\delta)|_{t=\delta} - v(\cdot,1)\|_m \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0, \tag{5.30}$$

uniformly in w_0, $(\rho_l)_{l\in\{1,\dots,N\}}$, and h from bounded subsets of H^{m+1}, $\mathrm{L}^2((0,1);\mathbb{R})$, *and* $\mathrm{L}^2((0,1);\mathrm{H}^{m-1})$ *respectively.*

Proof. When w_0 and $(\rho_l)_{l\in\{1,\dots,N\}}$ vary in bounded subsets of H^{m+1} and $\mathrm{L}^2((0,1);\mathbb{R})$ respectively, the solution v to the linear transport problem (5.29) remains in a bounded subset of $\mathrm{C}^0([0,1];\mathrm{H}^{m+1}) \cap \mathrm{W}^{1,2}((0,1);\mathrm{H}^m)$. With the aim of emphasizing that the convergence (5.30) will be uniform with respect to data from bounded subsets, we fix now any $M_0 > 0$ in a way that

$$w_0 \in \mathrm{B}_{\mathrm{H}^{m+1}}(0,M_0) := \left\{ f \in \mathrm{H}^{m+1} \mid \|f\|_{m+1} < M_0 \right\}.$$

Moreover, we select $w = S_\delta(w_0, h + \widetilde{\eta}_\delta, \widetilde{\aleph}_\delta)$ and denote the corresponding velocity field by $\boldsymbol{u} := \boldsymbol{\Upsilon}(w, \widetilde{\aleph}_\delta)$, where $S_\delta(\cdot,\cdot,\cdot)$ is the resolving operator on the time interval $[0,\delta]$ provided by Lemma 2.1 and $\boldsymbol{\Upsilon}(\cdot,\cdot)$ refers to the solution operator for the div-curl problem (2.4).

The central idea of the proof is to consider for any given $\delta \in (0,1)$ the velocity and vorticity expansions

$$w(\cdot,t) = v_\delta(\cdot,t) + r(\cdot,t), \quad \boldsymbol{u} = \overline{\boldsymbol{y}}_\delta + \boldsymbol{V}_\delta + \boldsymbol{R}, \quad v_\delta(\cdot,t) := v(\cdot,\delta^{-1}t), \tag{5.31}$$

where the remainder term $r\colon \mathbb{T}^2 \times [0,\delta] \longrightarrow \mathbb{R}$ is determined from w and v_δ, while the vector fields $\boldsymbol{R}$ and $\boldsymbol{V}_\delta$ are defined as

$$\boldsymbol{R} \coloneqq \boldsymbol{\Upsilon}(r, \mathbf{0}), \quad \boldsymbol{V}_\delta \coloneqq \boldsymbol{\Upsilon}(v_\delta, \boldsymbol{U}_\delta).$$

Due to the particular choice of $\widetilde{\boldsymbol{\aleph}}_\delta$, the ansatz for $\boldsymbol{u}$ in (5.31) is consistent with the definitions of $\overline{\boldsymbol{y}}_\delta$, $\boldsymbol{V}_\delta$, and $\boldsymbol{R}$. Furthermore, the initial condition $v(\cdot,0) = w_0$ implies that $r(\cdot,0) = 0$ in $\mathbb{T}^2$. Therefore, we can show (5.30) by verifying that

$$\|r(\cdot,\delta)\|_m \longrightarrow 0 \text{ as } \delta \longrightarrow 0. \tag{5.32}$$

The convergence in (5.32) shall be demonstrated during the following steps.

Step 1. Energy estimates for the remainder. After plugging the ansatz (5.31) and the definition of v_δ into the equation satisfied by $S_\delta(w_0, h + \widetilde{\eta}_\delta)$, which is of the form (2.15), the remainder r is seen to obey in $\mathbb{T}^2 \times (0,\delta)$ the initial value problem

$$\partial_t r - \nu\Delta r + ((\overline{\boldsymbol{y}}_\delta + \boldsymbol{V}_\delta + \boldsymbol{R}) \cdot \boldsymbol{\nabla})r + (\boldsymbol{R} \cdot \boldsymbol{\nabla})v_\delta = \xi_\delta, \quad r(\cdot,0) = 0, \tag{5.33}$$

with right-hand side $\xi_\delta \coloneqq h - (\boldsymbol{V}_\delta \cdot \boldsymbol{\nabla})v_\delta + \nu\Delta v_\delta$. In particular, for all $t \in [0,\delta]$ one has the bounds

$$\|\boldsymbol{R}(\cdot,t)\|_{m+1} \leq C_0\|r(\cdot,t)\|_m, \quad \|\boldsymbol{V}_\delta(\cdot,t)\|_{m+1} \leq C_0 \left(\|v_\delta(\cdot,t)\|_m + |\boldsymbol{U}_\delta(t)|\right), \tag{5.34}$$

where $C_0 > 0$ is the constant from (2.26).

We proceed by taking the $\mathrm{L}^2(\mathbb{T}^2)$-inner product of (5.33) with $\Delta^m r$, followed by integrating in time over the interval $[0,\delta]$. After integration by parts, while also applying the Poincaré inequality and noting that $\boldsymbol{\nabla} \cdot \boldsymbol{R} = 0$, one obtains for $t \in (0,\delta)$ the estimate

$$\begin{aligned}
&\|r(\cdot,t)\|_m^2 + 2\nu \int_0^t \|r(\cdot,s)\|_{m+1}^2 \, \mathrm{d}s \\
&\quad \leq C \int_0^t \|\xi_\delta(\cdot,s)\|_{m-1} \|r(\cdot,s)\|_{m+1} \, \mathrm{d}s \\
&\qquad + 2\int_0^t \|\boldsymbol{R}(\cdot,s)\|_{m+1} \left(\|v_\delta(\cdot,s)\|_{m+1}\|r(\cdot,s)\|_m + \|r(\cdot,s)\|_m\|r(\cdot,s)\|_{m+1}\right) \mathrm{d}s \\
&\qquad + 2\int_0^t \|\overline{\boldsymbol{y}}_\delta(s) + \boldsymbol{V}_\delta(\cdot,s)\|_{m+1}\|r(\cdot,s)\|_m^2 \, \mathrm{d}s \\
&\quad = I_1 + I_2 + I_3.
\end{aligned}$$

Regarding I_1, integration by parts, combined with inequalities of Young and Cauchy-Schwarz, yields for a constant $C = C(\nu) > 0$ the estimate

$$I_1 \leq C \int_0^t \|\xi_\delta(\cdot,s)\|_{m-1}^2 \, \mathrm{d}s + \frac{\nu}{4} \int_0^t \|r(\cdot,s)\|_{m+1}^2 \, \mathrm{d}s,$$

hence

$$I_1 \leq C \int_0^t \left(\|h(\cdot,s)\|_{m-1}^2 + \|(\boldsymbol{V}_\delta(\cdot,s)\cdot\boldsymbol{\nabla})v_\delta(\cdot,s)\|_{m-1}^2 + \|\nu\Delta v_\delta\|_{m-1}^2 \right) \mathrm{d}s \\ + \frac{\nu}{4}\int_0^t \|r(\cdot,s)\|_{m+1}^2 \,\mathrm{d}s. \tag{5.35}$$

Consequently, by a change of variables in the time integrals of (5.35) and taking into account that $t \in (0,\delta)$, one has the bound

$$I_1 \leq C \int_0^\delta \|h(\cdot,s)\|_{m-1}^2 \,\mathrm{d}s + C\delta \int_0^1 \|(\boldsymbol{V}_\delta(\cdot,\delta s)\cdot\boldsymbol{\nabla})v(\cdot,s)\|_{m-1}^2 \,\mathrm{d}s \\ + \delta\nu^2 C \int_0^1 \|v\|_{m+1}^2 \,\mathrm{d}s + \frac{\nu}{4}\int_0^t \|r(\cdot,s)\|_{m+1}^2 \,\mathrm{d}s.$$

Next, by utilizing Young's inequality and the estimate for $\boldsymbol{R}$ from (5.34), the integral I_2 is bounded via

$$I_2 \leq 2C_0 \int_0^t \|r(\cdot,s)\|_m \left(\|v_\delta(\cdot,s)\|_{m+1}\|r(\cdot,s)\|_m + \|r(\cdot,s)\|_m\|r(\cdot,s)\|_{m+1}\right) \mathrm{d}s \\ \leq C_0 \int_0^t \|v_\delta(\cdot,s)\|_{m+1}^2 \,\mathrm{d}s + \frac{\nu}{4}\int_0^t \|r(\cdot,s)\|_{m+1}^2 \,\mathrm{d}s + C\int_0^t \|r(\cdot,s)\|_m^4 \,\mathrm{d}s.$$

Furthermore, a change of variables implies

$$\int_0^t \|v_\delta(\cdot,s)\|_{m+1}^2 \,\mathrm{d}s \leq \delta \int_0^1 \|v(\cdot,s)\|_{m+1}^2 \,\mathrm{d}s \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0.$$

The integral I_3 shall be treated below in combination with a Grönwall argument. As a preparation, and by denoting the constant $M := \sup_{s\in[0,1]} \|\overline{\boldsymbol{y}}(s)\|_0$, which is independent of $\delta \in (0,1)$, we perform a change of variables in order to get

$$\widetilde{I}_{3,\delta}(t) := \int_0^t \|\overline{\boldsymbol{y}}_\delta(s) + \boldsymbol{V}_\delta(\cdot,s)\|_{m+1} \,\mathrm{d}s \\ \leq \int_0^1 (M + \delta C_0\|v(\cdot,s)\|_m + \delta C_0|\boldsymbol{U}(s)|) \,\mathrm{d}s \longrightarrow M \quad \text{as } \delta \longrightarrow 0.$$

Step 2. Conclusion. Let us collect the estimates related to I_1, I_2, and I_3 from the previous step. By resorting to Gönwall's inequality, for each $\delta \in (0,1)$ there exists a number $\varepsilon_\delta = \varepsilon_\delta(M_0) > 0$, which is independent of $t \in [0,1]$ and $w_0 \in \mathrm{B}_{\mathrm{H}^{m+1}}(0,M_0)$, such that $\varepsilon_\delta \longrightarrow 0$ for $\delta \longrightarrow 0$ and

$$\|r(\cdot,t)\|_m^2 \leq \left(\varepsilon_\delta + C\int_0^t \|r(\cdot,s)\|_m^4 \,\mathrm{d}s\right) \exp\left(\widetilde{I}_{3,\delta}(t)\right),$$

where $C > 0$ is independent of δ. Thus, there exists a new constant $C > 0$, which is likewise independent of $\delta \in (0,1)$, $t \in [0,\delta]$ and $w_0 \in \mathrm{B}_{\mathrm{H}^{m+1}}(0, M_0)$, such that

$$\|r(\cdot,t)\|_m^2 \leq C\varepsilon_\delta + C\int_0^t \|r(\cdot,s)\|_m^4 \,\mathrm{d}s =: \Theta(t).$$

As a result, by comparison with the differential inequality $\frac{\mathrm{d}}{\mathrm{d}t}\Theta \leq C\Theta^2$ (*cf.* [17, Proposition 2.2]), we conclude (5.32) and thus arrive at (5.30). □

For showing Theorem 2.3, it remains to combine the previously established results as follows.

Theorem 5.6. *Let $m \in \mathbb{N}$, $h \in \mathrm{L}^2((0,1);\mathrm{H}^{m-1})$, and $w_0, w_1 \in \mathrm{H}^{m+1}$ be fixed and denote for each $\delta > 0$ the average*

$$\widetilde{\aleph}_\delta(t) := \delta^{-1}\int_{\mathbb{T}^2} \overline{\boldsymbol{y}}(\delta^{-1}t)\,\mathrm{d}\boldsymbol{x}.$$

There are controls $(\zeta_{\boldsymbol{\ell}}^s, \zeta_{\boldsymbol{\ell}}^c)_{\boldsymbol{\ell}\in\mathcal{K}} \subset \mathrm{L}^2((0,1);\mathbb{R})$ such that

$$S_\delta\left(w_0, h + \delta^{-1}\widetilde{\eta}_{(\zeta_{\boldsymbol{\ell}}^s,\zeta_{\boldsymbol{\ell}}^c)_{\boldsymbol{\ell}\in\mathcal{K}}}(\cdot,\delta^{-1}\cdot), \widetilde{\aleph}_\delta\right)\Big|_{t=\delta} \longrightarrow w_1 \text{ in } \mathrm{H}^m \text{ as } \delta \longrightarrow 0,$$

uniformly with respect to the states w_0, w_1, and force h from bounded subsets of H^{m+1} and $\mathrm{L}^2((0,1);\mathrm{H}^{m-1})$ respectively.

Proof. Let $\widetilde{v}$ be the solution in $\mathbb{T}^2 \times (0,1)$ to the linear homogeneous transport equation

$$\partial_t\widetilde{v} + (\overline{\boldsymbol{y}}\cdot\boldsymbol{\nabla})\widetilde{v} = 0, \quad \widetilde{v}(\cdot,0) = w_0.$$

In particular, we know that $\widetilde{v}(1) = w_0$, as the flow $\boldsymbol{\mathcal{Y}}$ associated with the spatially constant vector field $\overline{\boldsymbol{y}}$ satisfies the property P2 from Section 3.3. Now, we fix any $\varepsilon > 0$ and apply Theorem 5.3 with the final state $\widehat{v}_1 := w_1 - w_0$, in order to obtain controls $(\zeta_{\boldsymbol{\ell}}^s, \zeta_{\boldsymbol{\ell}}^c)_{\boldsymbol{\ell}\in\mathcal{K}} \subset \mathrm{L}^2((0,1);\mathbb{R})$ such that the corresponding solution

$$\widehat{v} \in \mathrm{C}^0([0,1];\mathrm{H}^{m+1}) \cap \mathrm{W}^{1,2}((0,1);\mathrm{H}^m)$$

to the transport equation (5.16) obeys

$$\|\widehat{v}(\cdot,1) - \widehat{v}_1\|_{m+1} < \varepsilon.$$

Accordingly, the superposition $v := \widetilde{v} + \widehat{v}$ solves an initial value problem of the type (5.29) with right-hand side $\widetilde{\eta}_{(\zeta_{\boldsymbol{\ell}}^s,\zeta_{\boldsymbol{\ell}}^c)_{\boldsymbol{\ell}\in\mathcal{K}}}$ and fulfills the terminal condition

$$\|v(\cdot,1) - w_1\|_{m+1} < \varepsilon.$$

Resorting to Lemma 5.5 provides for sufficiently small $\delta > 0$ the estimate

$$\|S_\delta\left(w_0, h + \delta^{-1}\widetilde{\eta}_{(\zeta_{\boldsymbol{\ell}}^s,\zeta_{\boldsymbol{\ell}}^c)_{\boldsymbol{\ell}\in\mathcal{K}}}(\cdot,\delta^{-1}\cdot), \widetilde{\aleph}_\delta\right)\Big|_{t=\delta} - w_1\|_m < \varepsilon.$$

As explained in Remark 5.2, the family $(\zeta_\ell^s, \zeta_\ell^c)_{\ell\in\mathcal{K}}$ can be recovered from the initial and target states by means of a continuous linear operator. Hence, the controls $(\zeta_\ell^s, \zeta_\ell^c)_{\ell\in\mathcal{K}}$ remain in a bounded subset of $\mathrm{L}^2((0,1);\mathbb{R})$ when w_0 and w_1 vary in a bounded subset of H^{m+1}. Consequently, Lemma 5.5 allows choosing δ uniformly with respect to w_0, w_1 and h from respective bounded subsets of H^{m+1} and $\mathrm{L}^2((0,1);\mathrm{H}^{m-1})$. □

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A Curl-free vector fields supported near smooth cuts

The following theorem is likely known. As we could not locate the desired statement in the literature, the details are here provided in form of explicit constructions.

Theorem A.1. *Let $\Omega \subset \mathbb{T}^2$ be a subdomain containing smooth cuts $C_1, C_2 \subset \Omega$ such that $\mathbb{T}^2 \setminus (C_1 \cup C_2)$ is simply-connected. There exist $\mathbf{\Lambda}, \mathbf{\Sigma} \in \mathrm{C}^\infty(\mathbb{T}^2;\mathbb{R}^2)$ satisfying*

$$\mathbb{R}^2 = \operatorname{span}_{\mathbb{R}}\left\{\int_{\mathbb{T}^2} \mathbf{\Lambda}\, \mathrm{d}\boldsymbol{x}, \int_{\mathbb{T}^2} \mathbf{\Sigma}\, \mathrm{d}\boldsymbol{x}\right\}, \ \boldsymbol{\nabla} \wedge \mathbf{\Lambda} = \boldsymbol{\nabla} \wedge \mathbf{\Sigma} = 0, \ \operatorname{supp}(\mathbf{\Lambda}) \cup \operatorname{supp}(\mathbf{\Sigma}) \subset \Omega.$$

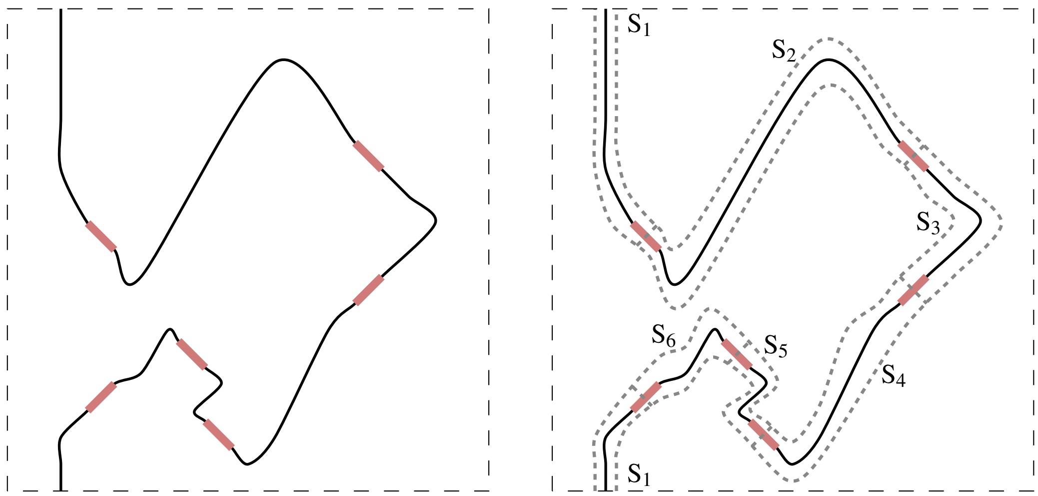

Figure 8: An illustration of the cut C_1. The displayed tubular neighborhood of C_1 comprises six sections $\mathrm{S}_1, \dots, \mathrm{S}_6$ such that C_1 is for $k \in \{1,2,3\}$ a graph over the x_1-axis in S_{2k} and a graph over the x_2-axis in S_{2k-1}. Near the internal section boundaries, the curve has the slope 1 or -1.

Proof. Since Ω is open, we may re-choose the smooth cuts $(C_i)_{i\in\{1,2\}}$ in a way that $\mathbb{T}^2 \setminus (C_1 \cup C_2)$ is simply-connected and the following properties hold.

- The cut C_1 equals the straight line $x_1 = c_1$ in $\mathbb{T}^1 \times ([0, r_1] \setminus (r_1/3, 2r_1/3))$, for some constant $c_1 \in \mathbb{T}^1$ and small $r_1 > 0$. Similarly, the cut C_2 equals the line $x_2 = c_2$ in $([0, r_2] \setminus (r_2/3, 2r_2/3)) \times \mathbb{T}^1$ for some $c_2 \in \mathbb{T}^1$ and small $r_2 > 0$.
- There exists a tubular neighborhood $\mathrm{N}(C_i) = \bigcup_{k=1}^{l_i} \mathrm{S}_k(C_i)$ of C_i with disjoint sections $\mathrm{S}_1(C_i), \dots \mathrm{S}_{l_i}(C_i)$, in a way that C_i is in each $\mathrm{S}_{2k+1-i}(C_i)$ a graph over the x_1-axis and in $\mathrm{S}_{2k-2+i}(C_i)$ a graph over the x_2-axis.
- The intersection $\mathrm{R}_{l,k}(C_i) \coloneqq \partial \mathrm{S}_l(C_i) \cap \partial \mathrm{S}_k(C_i)$ of two adjacent sections $\mathrm{S}_l(C_i)$ and $\mathrm{S}_k(C_i)$ is a single line segment with slope either 1 or -1. In the vicinity of the square with diagonal $\mathrm{R}_{l,k}(C_i)$, the curve C_i equals a straight line segment $\mathcal{L}_{l,k}(C_i) \subset C_i$, with slope either -1 or 1, such that the line $\mathrm{R}_{l,k}(C_i)$ passes transversely through $\mathcal{L}_{l,k}(C_i)$.

For the sake of a concise presentation, we assume that C_1 is the curve displayed in Figure 8 and only construct the vector field $\mathbf{\Lambda}$. The vector field $\mathbf{\Sigma}$ can be built in the same manner. In particular, that $\mathbf{\Lambda}$ and $\mathbf{\Sigma}$ have linearly independent averages will turn out as a generic property. Our treatment of the example in Figure 8 provides all the building blocks required for considering any general region Ω meeting the hypotheses of Theorem A.1.

Step 1. Constructions. Since we consider here only the curve C_1, let us fix for each $k \in \{1, \dots, l_1 = 6\}$ the names

$$\mathrm{N} = \mathrm{N}(C_1), \quad \mathrm{S}_k = \mathrm{S}_k(C_1).$$

Then, by further reducing the diameter of the tube N if necessary, we select six smooth functions $\widetilde{\upsilon}_k, \widehat{\upsilon}_k, \colon \mathbb{T}^1 \longrightarrow \mathbb{R}_-$, $k \in \{1,2,3\}$, having the below listed attributes.

- In a neighborhood N_{2k-1} of the section S_{2k-1}, satisfying $\overline{\mathrm{S}_{2k-1}} \subset \mathrm{N}_{2k-1}$, it holds that $x_1 + \widetilde{\upsilon}_k(x_2) = 0$ if and only if $(x_1,x_2) \in C_1$. When $(x_1,x_2) \in \mathrm{N}_{2k-1}$ is located right to C_1, then $x_1 + \widetilde{\upsilon}_k(x_2) > 0$. If $(x_1,x_2) \in \mathrm{N}_{2k-1}$ is located left to C_1, then $x_1 + \widetilde{\upsilon}_k(x_2) < 0$.
- In a neighborhood N_{2k} of section S_{2k}, satisfying $\overline{S}_{2k} \subset \mathrm{N}_{2k}$, it holds that $x_2 + \widehat{\upsilon}_k(x_1) = 0$ if and only if $(x_1,x_2) \in C_1$. When $(x_1,x_2) \in \mathrm{N}_{2k}$ is located above C_1, then $x_2 + \widehat{\upsilon}_k(x_1) > 0$. If $(x_1,x_2) \in \mathrm{N}_{2k}$ is located below C_1, then $x_2 + \widehat{\upsilon}_k(x_1) < 0$.

Furthermore, for a sufficiently small number $l \in (0, \operatorname{dist}(C_1, \partial\Omega))$, which will be determined later, we choose a cutoff $\beta \in \mathrm{C}^\infty(\mathbb{T}^1; \mathbb{R}_+)$ obeying

$$\operatorname{supp}(\beta) \subset (-l/2, l/2), \quad \beta(0) > 0. \tag{A.1}$$

Let us introduce the four main building blocks. Namely, for any general function $\upsilon \colon \mathbb{T}^1 \longrightarrow \mathbb{R}_-$ and $\boldsymbol{x} = [x_1,x_2]^\top \in \mathbb{T}^2$, we define

$$\widetilde{\mathbf{\Lambda}}^{\upsilon,\pm}(\boldsymbol{x}) := \begin{bmatrix} \pm\beta(\pm x_1 \pm \upsilon(x_2)) \\ \pm\beta(\pm x_1 \pm \upsilon(x_2))\frac{\mathrm{d}\upsilon}{\mathrm{d}s}(x_2) \end{bmatrix}, \; \widehat{\mathbf{\Lambda}}^{\upsilon,\pm}(\boldsymbol{x}) := \begin{bmatrix} \pm\beta(\pm x_2 \pm \upsilon(x_1))\frac{\mathrm{d}\upsilon}{\mathrm{d}s}(x_1) \\ \pm\beta(\pm x_2 \pm \upsilon(x_1)) \end{bmatrix}.$$

Then, we build $\mathbf{\Lambda}$ by gluing the previously introduced functions in a suitable way, resulting in the explicit formula

$$\mathbf{\Lambda}(x_1,x_2) := \begin{cases} \widetilde{\mathbf{\Lambda}}_k(x_1,x_2) & \text{if } (x_1,x_2) \in \mathrm{S}_{2k-1}, \\ \widehat{\mathbf{\Lambda}}_k(x_1,x_2) & \text{if } (x_1,x_2) \in \mathrm{S}_{2k}, \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

where the smooth vector fields

$$(\widetilde{\mathbf{\Lambda}}_k \colon \mathrm{N}_{2k-1} \longrightarrow \mathbb{T}^2)_{k\in\{1,2,3\}}, \quad (\widehat{\mathbf{\Lambda}}_k \colon \mathrm{N}_{2k} \longrightarrow \mathbb{T}^2)_{k\in\{1,2,3\}}$$

are given by

$$\widetilde{\mathbf{\Lambda}}_1(\boldsymbol{x}) = \widetilde{\mathbf{\Lambda}}^{\mathfrak{v}_1,+}, \qquad \widetilde{\mathbf{\Lambda}}_2(\boldsymbol{x}) = \widetilde{\mathbf{\Lambda}}^{\mathfrak{v}_2,+}, \qquad \widetilde{\mathbf{\Lambda}}_3(\boldsymbol{x}) = \widetilde{\mathbf{\Lambda}}^{\mathfrak{v}_3,-},$$
$$\widehat{\mathbf{\Lambda}}_1(\boldsymbol{x}) = \widehat{\mathbf{\Lambda}}^{\mathfrak{v}_1,+}, \qquad \widehat{\mathbf{\Lambda}}_2(\boldsymbol{x}) = \widehat{\mathbf{\Lambda}}^{\mathfrak{v}_2,-}, \qquad \widehat{\mathbf{\Lambda}}_3(\boldsymbol{x}) = \widehat{\mathbf{\Lambda}}^{\mathfrak{v}_3,-}.$$

The small parameter $l > 0$ is fixed in a way that $\widetilde{\mathbf{\Lambda}}_k$ and $\widehat{\mathbf{\Lambda}}_k$ are for each $k \in \{1, 2, 3\}$ supported in a neighborhood of C_1 which is sufficiently thin to ensure that $\mathbf{\Lambda}$ is a well-defined, smooth, and curl-free vector field obeying $\operatorname{supp}(\mathbf{\Lambda}) \subset \Omega$.

Step 2. Checking the average. It remains to study the average of $\mathbf{\Lambda} = [\Lambda_1, \Lambda_2]^\top$. To this end, we write $\mathrm{U} := \mathbb{T}^1 \times (0, r_1)$ and decompose

$$\int_{\mathbb{T}^2} \mathbf{\Lambda}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = \int_{\mathrm{U}} \mathbf{\Lambda}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{T}^2 \setminus \mathrm{U}} \mathbf{\Lambda}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}.$$

Due to (A.1) and Fubini's theorem, or alternatively by virtue of the Divergence Theorem, we arrive at

$$\int_{\mathrm{U}} \Lambda_1(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} > 0, \qquad \int_{\mathrm{U}} \Lambda_2(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = 0.$$

Therefore, by slightly perturbing C_1 in $\mathbb{T}^1 \times (r_1/3, 2r_1/3)$, one can change the value of $\fint_{\mathbb{T}^2} \Lambda_1(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}$ without affecting $\fint_{\mathbb{T}^2} \Lambda_2(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}$. The vector field $\mathbf{\Sigma} = [\Sigma_1, \Sigma_2]^\top$ is then obtained by analogous constructions, but now along the smooth cut C_2. Therefore, one can modify $\fint_{\mathbb{T}^2} \Sigma_2(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}$ in $(r_2/3, 2r_2/3) \times \mathbb{T}^1$ without impacting the value of the integral $\fint_{\mathbb{T}^2} \Sigma_1(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}$. In conclusion, we can first construct candidates for $\mathbf{\Lambda}$ and $\mathbf{\Sigma}$, followed by performing slight perturbations, if necessary, so that their averages are rendered linearly independent. □